Econ 702
Problem Set 2
Suggested Answers

Problem 1 In this problem the agent has two different ways to transfer resources from the present to the future: either by saving capital, or by buying state contigent claims. As we did in class, consider the following "experiment": if the agent sacrifices one unit of consumption and uses it as capital in the next period his return will be $R_{t+1}(h_t, z_{t+1})$, conditional on the realization of the shock in period $t+1$. So, for example, if $z_{t+1} = z^m$, the return will be $R_{t+1}(h_t, z^m)$.

How can we replicate this return in terms of state contigent claims? Suppose that the agent buys the following quantities: $R_{t+1}(h_t, z^1)$ from the state 1 contigent claim, $R_{t+1}(h_t, z^2)$ from the state 2 contigent claim, ..., $R_{t+1}(h_t, z^M)$ from the state M contigent claim. Then, clearly, regardless of the state that occurs the return to the agent, by the two alternative ways of transfering resources will be the same. The last step is to make sure that the expenditure (or just the cost) of the two alternatives was also equal. Since we've been considering the case in which the agent has to sacrifice one unit of consumption, the arbitrage condition is

$$\sum_m q_{t+1}(h_t, z^m)\ R_{t+1}(h_t, z^m) = 1,\ m = 1, 2, ..., M.$$

Problem 2 The problem is given by:

$$\max_{c_t(h_t), k_{t+1}(h_t)} \sum_t \beta^t \sum_{h_t} \pi(h_t) u\left(c_t(h_t)\right)$$

$$s.t:\ c_t(h_t) + k_{t+1}(h_t) + \sum_{z_{t+1}} q_t(h_t, z_{t+1})$$
$$b_{t+1}(h_t, z_{t+1}) = w_t(h_t) + k_t(h_{t-1}) R_t(h_t) + b_t(h_{t-1}, z_t)$$

The Langrangian of this problem is given by

$$\mathcal{L} = \sum_t \beta^t \sum_{h_t} \pi(h_t) u\left(c_t(h_t)\right)$$

$$+ \sum_t \sum_{h_t} \lambda_t(h_t) \left[c_t(h_t) + k_{t+1}(h_t) + \sum_{z_{t+1}} q_t(h_t, z_{t+1}) b_{t+1}(h_t, z_{t+1}) - w_t(h_t) - k_t(h_{t-1}) R_t(h_t) - b_t(h_{t-1}, z_t) \right]$$

Take the first order conditions

$$\{c_t(h_t)\}: \ \beta^t \pi(h_t) u_c(c_t(h_t)) = \lambda_t(h_t) \quad (1)$$

$$\{b_{t+1}(h_t, z^i)\}: \ \lambda_t(h_t) \ q_t(h_t, z^i) = \ \lambda_{t+1}(h_t, z^i) \qquad (2)$$

Then, replacing the Langrangian multiplier from (1) into (2) gives

$$\beta^t \pi(h_t) u_c(c_t(h_t)) \ q_t(h_t, z^i) = \ \beta^{t+1} \pi(h_t, z^i) u_c(c_{t+1}(h_{t+1}))$$

With the random shock being Markovian, we know that $\pi(h_t, z^i) = \pi(h_t)\Gamma_{ji}$, where Γ_{ji} is the probability of state i occuring in period $t+1$ given that the current state is i. Using this observation in the above expression we get

$$\pi(h_t) u_c(c_t(h_t)) \ q_t(h_t, z^i) = \ \beta \backslash \pi(h_t) \Gamma_{ji} u_c(c_{t+1}(h_{t+1})) \text{ and so}$$

$$q_t(h_t, z^i) = \beta \Gamma_{ji} \tfrac{{}_{ji} u_c(c_{t+1}(h_{t+1}))}{u_c(c_t(h_t))}$$

Finally, let's use the particular functional form that we have for prefernces, $u(c) = \frac{1}{1-\sigma} c^{1-\sigma}$, so that $u_c = c^{-\sigma}$. Then, the expression for the price of the state contigent claim is

$$q_t(h_t, z^i) = \beta \Gamma_{ji} \tfrac{(c_t)^{-\sigma}}{(c_{t+1})^{-\sigma}} = \beta \Gamma_{ji} \left(\tfrac{c_{t+1}}{c_t} \right)^{\sigma} \text{,where } i, j = 1, 2, 3.$$

Problem 3 The Social Planner's Problem is given by

$$\max_{c_t(h_t), k_{t+1}(h_t)} \sum_t \beta^t \sum_{h_t} \pi(h_t) u\left(c_t(h_t)\right)$$

$$s.t: \quad c_t(h_t) + k_{t+1}(h_t) = z_t f\left(k_t(h_{t-1}), 1\right) - \delta k_t(h_{t-1})$$

We will solve the problem in its recursive formulation. In general, showing the equivalence between the recursive problem and the problem of chosing infinite sequences (the originel problem) is not easy. However, for the purposes of this excercise, you can take as given that the two problems are equivalent, and thus proceed with (the easier) recursive version. This is given by

$$V(k,z) = \max_{c,k'} \left[u(c) + \beta \sum_{m=1}^{M} \Gamma_{im} V(k', z^m) \right]$$

$$s.t: \quad c + k' = zf(k,1) - \delta k \quad or$$

$$V(k,z) = \max_{c,k'} \left[u(zf(k,1) - \delta k - k') + \beta \sum_{m=1}^{M} \Gamma_{im} V(k', z^m) \right]$$

The First Order condition with respect to k' is:

$$-u_c(c) + \beta \sum_{m=1}^{M} \Gamma_{im} \tfrac{\partial}{\partial k'} V(k', z^m) = 0$$

In order to obtain an expression for the partial derivative of the value function we derive the envelope condition. Suppose that the solution has the form $k' = g(k,z)$. Then,

$$\tfrac{\partial V(k,z)}{\partial k} = u_c(c) \left[zf_1(k,1) - \delta - \tfrac{\partial g(k,z)}{\partial k} \right] + \beta \sum_{m=1}^{M} \Gamma_{im} \tfrac{\partial}{\partial k'} V(k', z^m) \tfrac{\partial g(k,z)}{\partial k}$$

Now gather all the terms that contain $\frac{\partial g(k,z)}{\partial k}$. We have

$$\tfrac{\partial V(k,z)}{\partial k} = \tfrac{\partial g(k,z)}{\partial k} \left[-u_c(c) + \beta \sum_{m=1}^{M} \Gamma_{im} \tfrac{\partial}{\partial k'} V(k', z^m) \right] + u_c(c) \left[zf_1(k,1) - \delta \right] = u_c(c) \left[zf_1(k,1) - \delta \right]$$

where the last equality follows from the First Order condition. Then from the Benveniste-Scheinkman Theorem we have

$$u_c(c) = \beta \sum_{m=1}^{M} \Gamma_{im} u_c(c') \left[z^m f_1(k',1) - \delta \right] \qquad \text{(Euler Equation)}$$

Next, we want to characterize the Steady State of this economy. We know that the Steady State is a situation where $k' = g(k) = k$, and let this particular capital stock be denoted by k^S. Of course, in order to find the Steady State we have to set the random shock equal to its theoretical mean, $\bar{z}$. Moreover, by plugging k^S in the budjet constraint, it is clear that the consumption will also be constant. Denote it by c^S. Then, using the Euler Equation,

$$u_c(c^S) = \beta \sum_{m=1}^{M} \Gamma_{im} u_c(c^S) \left[\bar{z} f_1(k^S, 1) - \delta\right] \qquad or$$

$$u_c(c^S) = \beta u_c(c^S) \left[\bar{z} f_1(k^S, 1) - \delta\right] \sum_{m=1}^{M} \Gamma_{im} \qquad or$$

$$1 = \beta \left[\bar{z} f_1(k^S, 1) - \delta\right]$$

The last thing to do is to use the specific functional form for the production function, $f(k, 1) = k^\theta$, so that $f_1(k, 1) = \theta k^{\theta-1}$. Then,

$$1 = \beta \left[\bar{z}\theta \left(k^S\right)^{\theta-1} - \delta\right] \qquad \text{which gives}$$

$$k^S = \left(\tfrac{\beta\theta\bar{z}}{1+\beta\delta}\right)^{1/(1-\theta)}$$

Using the budget constraint we can also find that

$$c^S = \left(\tfrac{\beta\theta\bar{z}}{1+\beta\delta}\right)^{1/(1-\theta)} \left[\bar{z}^{1+\theta} \left(\tfrac{\beta\theta}{1+\beta\delta}\right)^{\theta} - (1+\delta)\right]$$

Problem 4 Example 1: Logarithmic preferences and Cobb-Douglas production function. The recursive version of this problem is

$$V(k) = \max_{k'} \left[\ln(c) + \beta V(k')\right]$$

$$s.t : c + k' = Ak^a \qquad or$$

$$V(k) = \max_{k'} \backslash \left[\ln(Ak^a - k') + \beta V(k')\right]$$

We know that the value function has the form $V(k) = E + F\ln(k)$, where E and F are constants to be determined. So we can write

$$E + F\ln(k) = \max_{k'} \backslash\left[\ln(Ak^a - k') + \beta E + \beta F\ln(k)\right]$$

This is a well behaved concave program and the First Order condition will be sufficient as well as necessary. Taking the FOC we can find that

$$k' = \tfrac{\beta F A}{1+\beta F}k^a \quad \textit{and so}$$

$$E + F\ln(k) = \ln(Ak^a - \tfrac{\beta F A}{1+\beta F}k^a) + \beta E + \beta F\ln(\tfrac{\beta F A}{1+\beta F}k^a) \quad \textit{or}$$

$$E + F\ln(k) = \ln\left(Ak^a\left(1 - \tfrac{\beta F}{1+\beta F}\right)\right) + \beta E + \beta F\ln(\tfrac{\beta F A}{1+\beta F}) + a\beta F\ln(k) \qquad (1)$$

So by the method of undetermined coefficients

$$F = a + a\beta F \ \text{ or } F = \tfrac{a}{1-a\beta}$$

Plug the value of F in (1) and after some algebra you can find that

$$E = (1-\beta)^{-1}\left[\ln\left(A\left(1-\beta a\right)\right) + \tfrac{a\beta}{1-a\beta}\ln\left(Aa\beta\right)\right] \quad \textit{so that}$$

$$V(k) = (1-\beta)^{-1}\left[\ln\left(A\left(1-\beta a\right)\right) + \tfrac{a\beta}{1-a\beta}\ln\left(Aa\beta\right)\right] + \tfrac{a}{1-a\beta}k$$

What about the policy function? Recall that

$$k' = \tfrac{\beta F A}{1+\beta F}k^a \ \text{ and so}$$

$$k' = \frac{\beta\frac{a}{1-a\beta}A}{1+\beta\frac{a}{1-a\beta}}k^a \quad \textit{or}$$

$$k' = a\beta(Ak^a) = g(k)$$

Hence, in this particular example, the optimal policy is to invest a fraction $a\beta$ of the total output and consume the rest.

Example 2: CRRA preferences and linear constraint. The recursive version is

$$V(A) = \max_{c,A'} \left[\tfrac{1}{1-a}c^{1-a} + \beta V(A')\right]$$

$$s.t: \quad A' = R(A-c) \quad or$$

$$V(A) = \max_{c} \left[\tfrac{1}{1-a}c^{1-a} + \beta V(R(A-c))\right]$$

Note that here, unlike the previous example, we replaced A', not c. Take the First Order condition with respect to consumption:

$$c^{-a} = \beta B(1-a)R^{1-a}(A-c)^{-a} \quad or$$

$$c\left[1 + \left(\beta B(1-a)\,R^{1-a}\right)^{-\frac{1}{a}}\right] = A\left(\beta B(1-a)\,R^{1-a}\right)^{-\frac{1}{a}} \qquad \text{or setting}$$
$$k \triangleq \left(\beta B(1-a)\,R^{1-a}\right)$$

$$c = \tfrac{k^{-\frac{1}{a}}}{1+k^{-\frac{1}{a}}}A$$

We will guess that the value function has the form $V(A) = BA^{1-a}$, so we can write

$$BA^{1-a} = \tfrac{1}{1-a}\left(\tfrac{k^{-\frac{1}{a}}}{1+k^{-\frac{1}{a}}}A\right)^{1-a} + \beta BR^{1-a}\left(1 - \tfrac{k^{-\frac{1}{a}}}{1+k^{-\frac{1}{a}}}\right)^{1-a} A^{1-a} \qquad or$$

$$B = \tfrac{1}{1-a}\left(\tfrac{k^{-\frac{1}{a}}}{1+k^{-\frac{1}{a}}}\right)^{1-a} + \beta BR^{1-a}\left(\tfrac{1}{1+k^{-\frac{1}{a}}}\right)^{1-a} \quad or$$

$$B = \left(\tfrac{1}{1+k^{-\frac{1}{a}}}\right)^{1-a}\left[\tfrac{1}{1-a}k^{\frac{a-1}{a}} + \beta BR^{1-a}\right] \quad (2)$$

But note that

$$k^{\frac{a-1}{a}} = \left[\left(\beta B(1-a)\,R^{1-a}\right)\right]^{\frac{a-1}{a}} \text{ and so (2) becomes}$$

$$B = \left(\tfrac{1}{1+k^{-\frac{1}{a}}}\right)^{1-a}\left[\tfrac{1}{1-a}\left[\left(\beta B(1-a)\,R^{1-a}\right)\right]^{\frac{a-1}{a}} + \beta BR^{1-a}\right] \quad or$$

$$B = \left(\tfrac{1}{1+k^{-\frac{1}{a}}}\right)^{1-a}\left[(1-a)^{-\frac{1}{a}}\beta^{-\frac{1}{a}}B^{-\frac{1}{a}}R^{\frac{-(1-a)}{a}} + 1\right]\beta BR^{1-a} \quad or$$

$$k^{-\frac{1}{a}} = \beta^{-\frac{1}{a}} R^{\frac{-(1-a)}{a}} - 1 \quad or$$

$$\left[\beta B(1-a)R^{1-a}\right]^{-\frac{1}{a}} = \beta^{-\frac{1}{a}} R^{\frac{-(1-a)}{a}} - 1 \text{ and after some algebra}$$

$$B = \frac{1}{1-a}\left[\frac{\beta^{-\frac{1}{a}} R^{\frac{-(1-a)}{a}} - 1}{\beta^{-\frac{1}{a}} R^{\frac{-(1-a)}{a}}}\right]^{-a} = \frac{1}{1-a}\left[1 - \beta^{\frac{1}{a}} R^{\frac{1-a}{a}}\right]^{-a}$$

So the value function is given by

$$V(A) = \frac{1}{1-a}\left[1 - \beta^{\frac{1}{a}} R^{\frac{1-a}{a}}\right]^{-a} A^{1-a}$$

In order to obtain a closed form solution for the decision rule, recall that $c = \frac{k^{-\frac{1}{a}}}{1+k^{-\frac{1}{a}}} A \triangleq \lambda A \quad (3)$.

Then note that

$$k^{-\frac{1}{a}} = \left[\left(\beta B(1-a)\, R^{1-a}\right)\right]^{-\frac{1}{a}} \text{ or after replacing } B \text{ and a few lines of algebra}$$

$$k^{-\frac{1}{a}} = \beta^{-\frac{1}{a}} R^{\frac{a-1}{a}} - 1 \text{ and so}$$

$$\lambda = \frac{\beta^{-\frac{1}{a}} R^{\frac{a-1}{a}} - 1}{\beta^{-\frac{1}{a}} R^{\frac{a-1}{a}}} \text{ which implies (from (3)) that}$$

$$c(A) = \left[1 - \beta^{\frac{1}{a}} R^{\frac{1-a}{a}}\right] A$$

Finally, use the budget constraint to obtain

$$A' = R(A-c) \quad = R\left(1 - 1 + \beta^{\frac{1}{a}} R^{\frac{1-a}{a}}\right) A \quad or$$

$$A' = g(A) = (\beta R)^{\frac{1}{a}} A$$

Problem 5 First to make sure T is a well defined operator we have to make sure that a solution to the maximization problem exists and unique. Sufficient conditions for these are a compact and convex choice set and an strictly concave and continuous objective function. We can bound the asset holdings of the agent with a natural debt limit and apropriate assumptions on technology such that $a' \in [\underline{a}, \overline{a}]$.This would imply a closed and bounded set for consumption if we make sure the price functions are bounded. Note that if Inada conditions are assumed about technology we have,

$$F(0,1)=0, \lim_{K\to\infty} F_1(K,1)=0, \lim_{K\to 0} F_1(K,1)=\infty$$

so as $K\to\infty$ the consumption is bounded below by $-\underline{a}$. To make sure we have an upper bound we have to assume $G^e(K)$ is a nicely behaved continuous function with the following property;

$$G^e(K)\neq 0 \quad \text{for any } K\neq 0$$

such that asset returns do not go to infinity. Given these assumptions with U being continuous,strictly concave,F being continuously differentiable and our asset space being convex, T is a well defined operator.
Now let $\mathcal{C}$ be the set of continuous and bounded real-valued functions on R^2. Define a mapping $T:\mathcal{C}\to\mathcal{C}$ as follows:

$$\begin{gathered} T\varphi = \max\left\{u\left(c\right)+\beta\varphi\left(K',a';G\right)\right\} \\ s.t:\ c+a'=w(K)+aR(K) \\ K'=G^e\left(K\right) \end{gathered}$$

T is a contraction mapping because $\forall\varphi,\psi\in\mathcal{C}$

$$\begin{gathered} T\varphi = u\left(g\left(K\right)\right)+\beta\varphi\left(G\left(K\right)\right) = \\ u\left(g\left(K\right)\right)+\beta\psi\left(G\left(K\right)\right)+\beta[\varphi\left(G\left(K\right)\right)-\psi\left(G\left(K\right)\right)] \leq \\ \leq \max_g\left\{u\left(g\left(K\right)\right)+\beta\psi\left(G\left(K\right)\right)\right\}+\beta\left\|\varphi-\psi\right\| = T\psi+\beta\left\|\varphi-\psi\right\| \end{gathered}$$

By a symmetric argument we can establish, $\|T\varphi - T\psi\|\leq\beta\left\|\varphi-\psi\right\|$. Therefore, T is a contraction.

We can show $(\mathcal{C},d)$ is a complete metric space. then by the Contraction Mapping theorem (or Banach fixed point theorem),

1) T has a unique fixed point: There is a unique $V\in\mathcal{C}$ such that $TV=V$.

2) The sequence defined by $V^{n+1}=TV^n$ will converge to the fixed point V for any initial condition V_0.

b) To get the desired properties of the resulting value function we have to make certain assumptions on the primitives. The continuity of the value function comes from the fact that continuity assumption on U and compactness of the our state space. Under these assumptions, the Theorem of the Maximum states that the resulting value function is continous. For boundedness we can assume that U bounded and it would suffice but as you will notice the most widely used functional forms for utility functions do not satisfy the boundedness property. In that case, making sure that our state space is compact will ensure boundedness if our value function is continuous. The other two useful characteristics we would want our value function to have is strict concavity and strict increasing properties. To show these are indeed true if we assume U strictly incresing and strictly concave, we can take advantage of the contraction property of the operator T. We can show that the space of increasing and concave functions are complete and closed. If T maps increasing and concave functions into increasing and concave functions, which is true under our assumptions on U, then the convergent sequence of functions our operator generates must have a limiting function that is both increasing and concave by closedness property. Furthermore if we can show that the operator T.maps increasing and concave functions to strictly increasing and strictly concave functions, which is the case with our assumptions on U , then our value function must be strictly increasing and strictly concave since $TV = V$.

For a rigorous discussion of these issues, see Randy's Homework.