Econ 702, Spring 2005
Problem Set 3
Suggested Answers

Problem 1 The production function is $Y = F(K,N) = K^{\theta}N^{1-\theta}$. Then, the payments to the production factors is given by

$$wN + rK = pF_2(K,N)N + pF_1(K,N)K = p(1-\theta)\left(\frac{K}{N}\right)^{\theta}N + p\theta\left(\frac{K}{N}\right)^{\theta-1}K = (1-\theta+\theta)\,pK^{\theta}N^{1-\theta} = pY,$$

where p is the price of output not normalized to 1 as we usually do.

The capital's share is

$$a_k = \frac{rK}{pK^{\theta}N^{1-\theta}} = \frac{pF_1(K,N)K}{pK^{\theta}N^{1-\theta}} = \frac{\theta\left(\frac{K}{N}\right)^{\theta-1}K}{K^{\theta}N^{1-\theta}} = \frac{\theta K^{\theta}N^{1-\theta}}{K^{\theta}N^{1-\theta}} = \theta$$

and similarly, the labor's share is

$$a_n = \frac{wN}{pK^{\theta}N^{1-\theta}} = 1-\theta$$

Consider the following stylized facts:

(1) Per capita income is increasing, (2) Per capita capital is increasing, (3) The real wage is increasing, (4) The interest rate remains constant. Note that,

$\frac{Y}{N} = \frac{K^{\theta}N^{1-\theta}}{N} = \left(\frac{K}{N}\right)^{\theta}$, so (1) and (2) are consistent with each other.

The real wage is given by

$\frac{w}{p} = \frac{p(1-\theta)\left(\frac{K}{N}\right)^{\theta}}{p} = (1-\theta)\left(\frac{K}{N}\right)^{\theta}$, which is again consistent with the fact (2).

Finally the interest rate is given by

$r = p\theta\left(\frac{K}{N}\right)^{\theta-1} = p\theta\left(\frac{N}{K}\right)^{1-\theta}$, and so a constant interest rate can be explained for suitable movement of prices.

Problem 2 The utility function of the agent is given by $u(c,n,N)$, with $u_2 < 0$ and $u_3 < 0$. The economic problem here is that the agent thinks he cannot affect the aggregate hours of work. Of course, in equilibrium $n = N$ since the agent is representative, but the agent cannot realize that. In terms of mathematics, this means that whenever we take First Order Conditions with respect to n , only the second argument of the utility function will be affected. We begin by writting down the recursive formulation of the probblem.

$$V(k,a;G,H) = \max_{c,a',n}\left[u\left(c,n,N\right) + \beta V(k',a';G,H)\right]$$

$$\begin{aligned} s.t: \ c + a' &= nw\left(k,N\right) + aR\left(k,N\right) \\ k' &= G(k) \\ N &= H(k) \end{aligned}$$

or equivalently

$$V(k,a;G,H) = \max_{a',n}\left[u\left(nw\left(k,N\right) + aR\left(k,N\right) - a',n,N\right) + \beta V(k',a';G,H)\right]$$

$$\begin{aligned} s.t: k' &= G(k) \\ N &= H(k) \end{aligned}$$

$$\begin{gathered} w\left(k,N\right) = F_2(k,N) \\ R\left(k,N\right) = F_1(k,N) \\ \text{(by the competitive firm's behaviour)} \end{gathered}$$

The first order conditions are:

$$\{a'\}: \quad -u_1\left(c,n,N\right) + \beta\frac{\partial V(k',a';G,H)}{\partial a'} = 0 \qquad (1)$$

$$\{n\}: \ u_1\left(c,n,N\right)w\left(k,N\right) + u_2\left(c,n,N\right) = 0 \qquad (2)$$

We can be more precise about (1) by using the envelope condition. Suppose that the solution has the form, $a' = g\left(k,a;G,H\right)$ and $n = h\left(k,a;G,H\right)$. Then,

$$\frac{\partial V(k,a;G,H)}{\partial a} =$$
$$u_1(c,n,N)\left[w(k,N)\frac{\partial h}{\partial a} + R(k,N) - \frac{\partial g}{\partial a}\right] + u_2(c,n,N)\frac{\partial h}{\partial a} + \frac{\partial V(k',a';G,H)}{\partial a'}\frac{\partial g}{\partial a} =$$

$$= \frac{\partial h}{\partial a}\left[u_1(c,n,N)\,w(k,N) + u_2(c,n,N)\right] + \frac{\partial g}{\partial a}\left[-u_1(c,n,N) + \beta\frac{\partial V(k',a';G,H)}{\partial a'}\right] +$$
$$u_1(c,n,N)\,R(k,N) = u_1(c,n,N)\,R(k,N)$$

where in the last equality we just used the First Order Conditions.

Summing up, the two equations that characterize the RCE are the following

$$u_1(c,n,N) = \beta \quad u_1(c',n',N')\,F_1(k',N') \qquad (1')$$

$$u_1(c,n,N)\,F_2(k,N) + u_2(c,n,N) = 0 \qquad (2)$$

Note: If we want to be precise, the two equations that characterize the RCE are the above, but after replacing the solutions and conditions that we have, so that everything is a function of the aggregate state only (here k). We will not do this here since our primar task is to show the discrepancy between this solution and the social planner's solution. See next problem for more details.

Let's now go to the Social Planner's Problem. The Social Planner does not care about prices, only about primitives (preferences and technology). The only state for the Planner is k. Note that the utility function that will appear in this problem is exactly the same (otherwise the two cases wouldn't be comparable), but the Planner realizes that $n = N$. We will assume that the control variable is n, but according to the above comment $dN/dn = 1$.(So the variable N will only appear in the utility function in order to highlight the externality). The problem is given recursively by

$$V(k) = \max_{k',n}\left[\ u(c,n,N) + \beta V(k')\right]$$

$$s.t: \quad c + k' = F(k,n)$$
$$and \quad N = n$$

The First Order Conditions are:

$$\{k'\}: \quad -u_1(c,n,N) + \beta\frac{dV(k')}{dk'} = 0 \qquad (S1)$$

$$\{n\}: \ u_1(c,n,N)\,F_2(k,N) + u_2(c,n,N) + u_3(c,n,N)\frac{dN}{dn} = 0$$

$$\text{or} \quad u_1(c,n,N)\,F_2(k,N) + u_2(c,n,N) + u_3(c,n,N) = 0 \qquad (S2)$$

Let's use the envelope condition in order to obtain an expression for the derivative of the value function in (S1). To that end, suppose that the solution is of the form $k' = g(k), \quad n = h(k)$. Then,

$$\frac{dV(k)}{dk} = u_1(c,n,N)\left[-g'(k) + F_2(k,N)h'(k) + F_1(k,N)\right] + u_2(c,n,N)\,h'(k) + u_3(c,n,N)\,\frac{dN}{dn}h'(k) + \beta\frac{dV(k')}{dk'}g'(k) =$$

$$= g'(k)\left[-u_1(c,n,N) + \beta\frac{dV(k')}{dk'}\right] + \beta h'(k)\left[\, u_1(c,n,N)\,F_2(k,N) + u_2(c,n,N) + u_3(c,n,N)\right] + u_1(c,n,N)\,F_1(k,N) = u_1(c,n,N)\,F_1(k,N)$$

So for the Planner's problem we have:

$$u_1(c,n,N) = \beta u_1(c',n',N')\,F_1(k',N') \qquad (S1')$$

$$u_1(c,n,N)\,F_2(k,N) + u_2(c,n,N) + u_3(c,n,N) = 0 \qquad (S2)$$

Compare (2) and (S2). Clearly the solutions to the two problems do not coincide. Note that since $u_3 < 0$,

$$u_1(c^s,n^s,N^s)\,F_2(k^s,N^s) + u_2(c^s,n^s,N^s) > 0 = u_1\left(c^E,n^E,N^E\right)F_2(k^E,N^E) + u_2\left(c^E,n^E,N^E\right) \quad (3)$$

Economic intuition says that $N^s < N^E$, since hours of work produce a negative externality. Suppose, by way of contradiction, that $N^s > N^E$. Then,

$$u_2(c^s,n^s,N^s) < u_2\left(c^E,n^E,N^E\right) \quad (i)$$

Note that the first argument in the above partials is not the same. So we are implicitly assuming that $|u_{21}|$ is much smaller than $|u_{22}|$.
Moreover, we have $c^s > c^E$ and so

$$u_1(c^s,n^s,N^s) < u_1\left(c^E,n^E,N^E\right) \qquad (ii)$$

again assuming that $|u_{11}|$ is significantly greater than $|u_{12}|$.

Finally, from the concavity of F, and the contradictory assumption,

$$F_2(k^s, N^s) < F_2(k^E, N^E) \qquad (iii)$$

Multiplying (ii) by (iii) and adding (i) yields

$$\begin{gathered} u_1\left(c^s, n^s, N^s\right) F_2(k^s, N^s) + u_2\left(c^s, n^s, N^s\right) < \\ u_1\left(c^E, n^E, N^E\right) F_2(k^E, N^E) + u_2\left(c^E, n^E, N^E\right) \end{gathered}$$

a contradiction to (3)

So it has to be the case that $N^s < N^E$.

Problem 3 The problem is the following

$$V(k, a; G, H) = \max_{c, a', n} \left[u\left(c, n\right) + \beta V(k', a'; G, H)\right]$$

$$\begin{aligned} s.t: \quad c + a' &= nw\left(k, N\right)\left(1 - \tau\right) + a\left(R\left(k, N\right) - 1\right)\left(1 - \tau\right) + a + T \\ k' &= G(k) \\ N &= H(k) \end{aligned}$$

$$T = T(k) = \tau\left[Nw\left(k, N\right) + k\left(R\left(k, N\right) - 1\right)\right]$$

Note the small difference with your lecture notes: Here the tax revenues are not thrown into the ocean, they are returned to the agent. So T will be in the budget constraint and we have to define it (as is true for everything else) as a function of the aggregeta state.

We can replace the consumption from the budget constraint and write

$$\begin{gathered} V(k, a; G, H) = \\ \max_{a', n} \left[u\left(\ nw\left(k, N\right)\left(1 - \tau\right) + a\left(R\left(k, N\right) - 1\right)\left(1 - \tau\right) + a + T - a', n\right) + \beta V(k', a'; G, H)\right] \end{gathered}$$

$$\begin{aligned} s.t: k' &= G(k) \\ N &= H(k) \end{aligned}$$

$$T = T(k) = \tau\left[Nw\left(k, N\right) + k\left(R\left(k, N\right) - 1\right)\right]$$

The First Order Conditions for this problem are:

$$\{a'\}: \quad -u_1(c,n) + \beta \frac{\partial V(k',a';G,H)}{\partial a'} = 0 \qquad (1)$$

$$\{n\}: \ u_1(c,n)\, w(k,N)(1-\tau) + u_2(c,n) = 0 \qquad (2)$$

Suppose that the solutions have the form $a' = g(k,a;G,H)$ and $n = h(k,a;G,H)$. Then the envelope condition is just

$$\frac{\partial V(k,a;G,H)}{\partial a'} = u_1(c,n)\left[1 + (R(k,N)-1)(1-\tau)\right]$$

So the two conditions that characterize the RCE are

$$u_1(c,n) = \quad \beta u_1(c',n')\left[1 + (R(k',N')-1)(1-\tau)\right] \qquad (1')$$

$$u_1(c,n)\, w(k,N)(1-\tau) + u_2(c,n) = 0 \qquad (2)$$

and, clearly, the allocation they imply is not the same as without the distortionary tax.

In order to be more precise, we have to write everything as a function of the aggregate state. The two equations become (write $g(k,a;G,H)$=$g(k,a)$ and $h(k,a;G,H)$=$h(k,a)$for simplicity):

$$u_1\left[h(k,a)F_2(k,H(k))(1-\tau) + a(F_1(k,H(k))-1)(1-\tau) + a + T - g(k,a), h(k,a)\right] =$$

$$=$$

$$\beta u_1\left[h(G(k),g(k,a))F_2(G(k),H(G(k))(1-\tau) + g(k,a)\left(F_1(G(k)),H(G(k)))\right)-1)(1-\tau) - g(G(k),g(k,a)) +\right.$$

$$\left[1 + (F_1(G(k),H(G(k))-1)(1-\tau)\right]$$

and

$$u_1\left[h(k,a)F_2(k,H(k))(1-\tau) + a(F_1(k,H(k))-1)(1-\tau) + a + T - g(k,a), h(k,a)\right] F_2(k,H(k))(1-\tau)$$

$$+u_2\left[h(k,a)F_2(k,H(k))(1-\tau) + a(F_1(k,H(k))-1)(1-\tau) + a + T - g(k,a), h(k,a)\right] = 0$$

or, finally, by imposing the repesentative agent condition (which says that $g(k,k) = G(k)$ and $h(k,k) = H(k)$)

$$u_1 \left[H(k)F_2(k,H(k))\,(1-\tau) + k\,(F_1\,(k,H(k)) - 1)\,(1-\tau) + k + T - G(k), H(k)\right] =$$

$$=$$

$$\beta u_1 \left[H(G(k))F_2(G(k),H(G(k))\,(1-\tau) + G(k)\,(F_1\,(G\,(k))\,,H(G\,(k)))) - 1)\,(1-\tau) - G(G(k)) + G(k) + T', H(\right.$$

$$\left[1 + (F_1\,(G(k),H(G(k)) - 1)\,(1-\tau)\right]$$

and

$$u_1 \left[H(k)F_2(k,H(k))\,(1-\tau) + k\,(F_1\,(k,H(k)) - 1)\,(1-\tau) + k + T - G(k), H(k)\right] F_2\,(k,H(k))\,(1-\tau)$$

$$+u_2 \left[H(k)F_2(k,H(k))\,(1-\tau) + k\,(F_1\,(k,H(k)) - 1)\,(1-\tau) + k + T - G(k), H(k)\right] = 0$$

where

$$T = T(k) = \tau\left[H(k)F_2(k,H(k)) + k\,(F_1\,(k,H\,(k))) - 1)\right]$$

and

$$T' = \tau\left[H(G(k))(F_2(G(k),H(G(k))) + k\,(F_1\,(G(k),H\,(G(k)))) - 1)\right]$$

The above expressions are functional equations, where all functions depend on k only.

Problem 4 In this problem we have to come up with a tax scheme that will cancel out the effect of the negative externality described in problem 2. Recall from the social planner's problem that the (optiomal) allocation is characterized by the following equations:

$$u_1\,(c,n,N) = \beta u_1\,(c',n',N')\,F_1(k',N') \qquad (S1')$$

$$u_1\,(c,n,N)\,F_2(k,N) + u_2\,(c,n,N) + u_3\,(c,n,N) = 0 \qquad (S2)$$

Returning to the competitive equilibrium, suppose that tax rate on wage income is τ and the tax rate on capital income is t_c. We will set $t_c = 0$. Then the recursive version of the problem of the representative agent is:

$$V(k, a; G, H) = \max_{c,a',n} \left[u\left(c, n, N\right) + \beta V(k', a'; G, H)\right]$$

$$\begin{aligned} s.t: \quad c + a' &= nw\left(k, N\right)\left(1 - \tau\right) + aR\left(k, N\right) + T \\ k' &= G(k) \\ N &= H(k) \end{aligned}$$

$\tau = \tau\left(k\right)$; we'll see what form it has to have, and

$$T = T(k) = \tau\left[Nw\left(k, N\right)\right]$$

or equivalently

$$V(k, a; G, H) =$$
$$\max_{a',n} \left[u\left[c + a' = nw\left(k, N\right)\left(1 - \tau\right) + aR\left(k, N\right) + T, n, N\right] + \beta V(k', a'; G, H)\right]$$

$$\begin{aligned} s.t: k' &= G(k) \\ N &= H(k) \end{aligned}$$

$\tau = \tau\left(k\right)$; we'll see what form it has to have, and

$$T = T(k) = \tau(k)\left[Nw\left(k, N\right)\right]$$

The First Order Conditions for this problem are:

$$\{a'\}: \quad -u_1\left(c, n, N\right) + \beta \frac{\partial V(k', a'; G, H)}{\partial a'} = 0 \qquad (1)$$

$$\{n\}: \ u_1\left(c, n, N\right) w\left(k, N\right)\left(1 - \tau\right) + u_2\left(c, n, N\right) = 0 \qquad (2)$$

You should be confident with the use of the Envelope Condition by now. It is

$$\frac{\partial V(k, a; G, H)}{\partial a} = u_1\left(c, n, N\right) R\left(k, N\right) \text{ implying that}$$

$$u_1\left(c, n, N\right) = \beta \ u_1\left(c', n', N'\right) R\left(k', N'\right) \qquad (1')$$

Summing up (and using the firm's behaviour)

$$u_1\,(c,n,N)=\beta\; u_1\,(c',n',N')\,F_1\,(k',N') \qquad (1') \quad \text{and}$$

$$u_1\,(c,n,N)\,F_2\,(k,N)\,(1-\tau)+u_2\,(c,n,N)=0 \qquad (2)$$

characterize the RCE allocation.

Compare (1') and (S1'). It can be easily seen that they are identical (in the SPP n is aggregate labor). So if we could find a way to replicate (S2) by suitably chosing the tax ratem in (2), the competitive allocation would coincide with the Social Planner's, and hence it would be optimal. This can be achieved by setting

$$\tau(k)=-\tfrac{u_3(c,n,N)}{F_2(k,N)\;u_1(c,n,N)}$$

We saw in Problem 2 that in the competitive equilibrium people tend to work too much, because they can't internalize the externality. A positive wage income tax, like the one described above, can achieve this task (make people internalize the negative externality), and can thus lead to an efficient allocation of resources.

Problem 5 In this problem the goverment issues debt in order to pay the previous debt. The Goverment Budget Constraint is given by

$$(1+r_b)B+\bar{G} \;=\; B'+\tau\,[F(k,1)-k]$$

Now B will be a state and the problem of the agent is given by

$$V(k,B,a;G,H)=\max_{c,a'}\,[u\,(c)+\beta V(k',B',a';G,H)]$$

$$\begin{gathered} s.t: \quad c+a'=w\,(1-\tau)+a\,(R-1)\,(1-\tau)+a \\ k'=G(k,B) \\ B'=H(k,B) \end{gathered}$$

$$\begin{gathered} w=w\,(k)=F_2\,(k,1) \\ R=R\,(k)=F_1\,(k,1) \\ \tau=\tau\,(k,B) \end{gathered}$$

or equivalently

$$V(k,B,a;G,H)=$$
$$\max_{a'}\left[u\left(w\left(1-\tau\right)+a\left(R-1\right)\left(1-\tau\right)+a-a'\right)+\beta V(k',B',a';G,H)\right]$$

$$s.t: k'=G(k,B)$$
$$B'=H(k,B)$$

$$w=w\left(k\right)=F_2\left(k,1\right)$$
$$R=R\left(k\right)=F_1\left(k,1\right)$$
$$\tau=\tau\left(k,B\right)$$

A Recursive Competitive Equilibrium is a list of functions $\{V(k,B,a;G,H),\ g(k,B,a;G,H),\ G(k,B),\ H(k,$

such that

1) Given $G(k,B),\ H(k,B)$,$\tau\left(k,B\right)$ the functions $V(k,B,a;G,H),\ g(k,B,a;G,H)$

solve the agent's maximization problem.

2) The Goverment Budget Constraint is balanced

$$(1+r_b)B+\ \bar{G}\ =H(k,B)+\tau\left(k,B\right)\left[F(k,1)-k\right]$$

3) The following arbitrage condition holds

$$1+r_b=1+\left(F_1(k,1)-1\right)\left(1-\tau\left(k,B\right)\right)$$

4) Representative agent condition

$$g\left(k,B,k+B\right)=\ G\left(k,B\right)+H(k,B)$$

Claim: There has to be a limit in the sequence $\{B_t\}_{t=0}^{\infty}$.

Why? By (4) if $B_t\rightarrow\infty$, then $a_t\rightarrow\infty$. But then the transversality condition for the agent's maximization problem will be violated. Hence, a sequnce $\{B_t\}_{t=0}^{\infty}$ that increases in an unbounded way cannot be part of the RCE (since (1) is violated).

Summing up, a RCE is a list of functions $\{V(k,B,a;G,H),\ g(k,B,a;G,H),\ G(k,B),\ H(k,B),\tau\left(k,B\right)\}$

such that conditions (1) -(4) are satisfied, plus :

5) There exist $\underline{B}, \bar{B}$ and $\bar{k}$ such that for every $(k, B) \in \left[0, \bar{k}\right] \times \left[\underline{B}, \bar{B}\right]$, $G(k, B) \in \left[0, \bar{k}\right]$ and $H(k, B) \in \left[\underline{B}, \bar{B}\right]$.