Growth Model

Question 1

This is a two-sector growth model. There is an investment good sector and a consumption good sector.

Commodity Space:

$$L = \{x \mid s(h_t) = (s^A_{jt}(h_t), s^A_{jt}(h_t)) \in (R^2)^3 \ \forall t, h_t \text{ and } \|s\|_\infty < \infty\}$$

Consumption possibility set: —

$$X = \{x^A, x^B \in L : \exists \{c^A_t(h_t), c^B_t(h_t), k_{t+1}(h_t)\}_{t=0}^{\infty} \geq 0 \text{ such that}$$

$$\begin{aligned}
c^A_t(h_t) + k_{t+1}(h_t) &= x^A_{1t}(h_t) + (1-\delta)k_t(h_{t-1}) \qquad \forall t, \forall h_t \\
c^B_t(h_t) &= x^B_{1t}(h_t) \qquad \forall t, \forall h_t \\
x^A_{2t}(h_t) + x^B_{2t}(h_t) &\leq k_t(h_{t-1}) \qquad \forall t, \forall h_t \\
x^A_{3t}(h_t) + x^B_{3t}(h_t) &\in [0,1] \qquad \forall t, \forall h_t \\
& \quad k_0, h_0 \text{ given}\}
\end{aligned}$$

where $x^A_{1t}(h_t)$=output goods in terms of apples, $x^A_{3t}(h_t)$=labor services supply to technology A, $x^A_{2t}(h_t)$=capital service to technology A, $x^B_{1t}(h_t)$=output goods in terms of bananas, $x^B_{2t}(h_t)$=labor services supply to technology B, $x^B_{3t}(h_t)$=capital service to technology B.

Production possibility set:

$$\begin{aligned}
Y^A &= \{y \in L : y^A_{1t}(h_t) \leq F^A(-y^A_{2t}(h_t), -y^A_{3t}(h_t)),\ y^A_{1t}(h_t) \geq 0, y^A_{2t}(h_t) \leq 0, y^A_{3t}(h_t) \leq 0\} \\
Y^B &= \{y \in L : y^B_{1t}(h_t) \leq F^B(-y^B_{2t}(h_t), -y^B_{3t}(h_t)),\ y^B_{1t}(h_t) \geq 0, y^B_{2t}(h_t) \leq 0, y^B_{3t}(h_t) \leq 0\}
\end{aligned}$$

Consumer's problem:

$$\max \sum_t \sum_{h_t} u[c^A_t(h_t), c^B_t(h_t), 1 - x^A_{2t}(h_t) + 1 - x^B_{2t}(h_t)]$$

$$\text{s.t.} \qquad \sum_t \sum_{h_t} \sum_{i=1}^{3} [p^A_{it}(h_t) x^A_{it}(h_t) + p^B_{it}(h_t) x^B_{it}(h_t)]$$

Producers' problem:

$$\max_{y^A \in Y^A} \sum_{i=1}^{3} [p^A_{it}(h_t) y^A_{it}(h_t)]$$

$$\max_{y^B \in Y^B} \sum_{i=1}^{3} [p^B_{it}(h_t) y^B_{it}(h_t)]$$

An Arrow-Debreu Competitive Equilibrium is $(p^{A^*}, x^{A^*}, y^{A^*}, p^{B^*}, x^{B^*}, y^{B^*})$ such that

1. x^{A^*}, x^{B^*} solves the consumer's problem.
2. y^{A^*}, y^{B^*} solves the firm's problem.
3. Markets clear,

$$\begin{aligned} x_{1t}^{i*}(h_t) &= y_{1t}^{*^i}(h_t)\ i = A, B \text{ for all } h_t, t \\ x_{2t}^{i*}(h_t) &= -y_{2t}^{*^i}(h_t)\ i = A, B \text{ for all } h_t, t \\ x_{3t}^{i*}(h_t) &= -y_{3t}^{*^i}(h_t)\ i = A, B \text{ for all } h_t, t \end{aligned}$$

Question 2

Theorem 1 *(FBWT) If the preferences of consumers are nonsatiated ($\exists\{x_n\} \in X$ that converges to $x \in X$ such that $U(x_n) > U(x)$), an allocation (x^*, y^*) of an ADE (p^*, x^*, y^*) is PO.*

Theorem 2 *(SBWT) If (i) X is convex, (ii) preference is convex (for $\forall x, x' \in X$, if $x' < x$, then $x' < (1-\theta)x' + \theta x$ for any $\theta \in (0,1)$), (iii) $U(x)$ is continuous, (iv) Y is convex, (v) Y has an interior point, then with any PO allocation (x^*, y^*) such that x^* is not a satuation point, there exists a continuous linear functional p^* such that (x^*, y^*, p^*) is a Quasi-Equilibrium ((a) for $x \in X$ which $U(x) \geq U(x^*)$ implies $p^*(x) \geq p^*(x^*)$ and (b) $y \in Y$ implies $p^*(y) \leq p^*(y^*)$)*

Question 3

First write down the problem of the household for future reference. The aggragate state variables are technology shock vector $z = (z^A, z^B)$ capital stock vector of two sectors K and individual state variable is asset holdings of the household. Note that the stochastic processes are independent. The problem in recursive formulation;

$$\begin{aligned} V(a, z, K) &= \max_{c^A, c^B, a', n} \left(\frac{\phi(c^A, c^B)^{1-\sigma}}{1-\sigma} + \alpha(1-n)\right) + \beta \sum_{z'} \Gamma^A \Gamma^B V(a', z', K') \\ s.t. \quad c^A + p^B c^B + a' &= wn + Ra \end{aligned}$$

given,

$$\begin{aligned} R &= z^A F_1(K^A, N^A) + 1 - \delta = p^B z^B F_1(K^B, N^B) + 1 - \delta \\ w &= z^A F_2(K^A, N^A) = z^B F_2(K^B, N^B) p^B \\ K &= G(K, z)\ ,\ \ N = H(K, z) \\ N^A &= \Phi(K, z),\ K^A = \Psi(K, z),\ p^B = p(z, K) \end{aligned}$$

and has the solution,

$$\begin{aligned} g(K,z,a) &= a' \\ h(K,z,a) &= n \\ c^A(K,z,a) &= c^A \\ c^B(K,z,a) &= c^B \end{aligned}$$

Note that we are in a representative agent environment and complete market setup is equivalent to closing markets down. Also the numeriare good is apples. Factor mobility ensures equal return on both sectors.

In question 3, You are asked for sufficient conditions (not unique) for apples not to be produced. Apples being the capital good face two margins one being banana-apple consumption other intertemporal margin for apples. We are looking for sufficient condition for a corner solution in which no apples produced. From FOC for a given K one can derive the following condition,

$$\lim_{N^A \to 0}[z^A F_2^A(K^A, N^A)] \leq \lim_{N^A \to 0}[\frac{\alpha}{Q_A(C_A(K^A, N^A), C_B(K^A, N^A))^{-\sigma}}]$$

and $F^A(K^A, 0) = 0$ for any $K^A \in [0, K]$.

Question 4

Definition of recursive equilibrium;

A RCE is a list of functions $\{V(.),\ G(.),\ H(.),\ g(.),\ h(.), c^i(.), \Phi(.),\ \Psi(.)\}$, $i \in \{A, B\}$ and $\{w(z,K),\ R(z,K), p(z,K)\}$ such that,

-(Household optimization)

Given $\{G(.),\ H(.), \Phi(.), \Psi(.)\}$and $\{w(z,K),\ R(z,K), p(z,K)\}$, $\{V(.),\ g(.), h(.), c^A(.), c^B(.)\}$ solves the households' problem

-(Aggragate Consistency)

$$h(z,K,K) = H(z,K)$$

$$g(z,K,K) = G(z,K)$$

-(Market Clearing)

$$\begin{aligned} G(K,z) &= z^A F^A(z,K) - C^A(z,K) + (1-\delta)K \\ C^B(z,K) &= F^B(K^B, N^B) \end{aligned}$$

Note that one of the consumption functions $c^A(.), c^B(.)$ is redundant from the budget constraint of the household here. The fact that it is written down as a part of the definition means, budget constraint is not used for this purpose and free to be used in the rest of the definition. This is implicit from the market clearing conditions in which both market clearing conditions are explicitly part of the definition and makes the budget constraint redundant through Walras' law (only 2 of the 3 equations (2 market clearing and 1 budget constraint), are non-redundant). A more conventional way (the way its done usually in class) of defining this equilibrium would be to leave out one of the consumption functions (using the budget constraint), and use only one of the market clearing conditions, which leaves us one less equilibrium object and one less condition. It is a useful way to think of the equilibrium as the solution to a set of functional equations but as Victor warned in class, do not get the impression that each particular condition pins down a particular equilibrium function since most of the times they are determined simultaneously. The important thing here is to realize that there is no unique way of defining an equlibrium as long as the definition is consistent.

Question 5

Note that there are no externalities or such we can use the SPP allocation to get prices. When both produced we can get the relative price formula from FOC of the firms' problem as an interior solution,

$$p^B = \frac{z^A F_2^A(K^{*A}, N^{*A})}{z^B F_2^B(K^{*B}, N^{*B})}$$

where K^{*i}, N^{*i} are SPP allocations. If no apples are produced, we can no longer use the formula above, instead the price can be obtained from the margin at consumption;

$$p^B = \left(\frac{Q_1(C^{*A}, C^{*B})}{Q_2(C^{*A}, C^{*B})} \right)^{-\sigma}$$

Question 6

With the introduction of taxes and subsidies the budget constraint becomes

$$c^A + p^B(z,K)(1+\tau(z,K))c^B + a' = [w(z,K) + \theta(z,K)]n + R(z,K)a$$

and assuming period by period balanced budget for government government can only choose $\tau(z,K)$ or $\theta(z,K)$ and let the other determined from its budget constraint.

$$N^*\theta(z,K) = p^B(z,K)(1+\tau(z,K))C^{*B}$$

where $N^* = h(z,K,K) = H(z,K)$ and C^{*B} is equilibrium level of banana consumption given a feasible government policy. This is the formula linking the equilibrium values of these functions.

1 Lucas Trees

Recursive formulation of the problem; state variables are two stochastic shocks $(d_i, z_j)$and share of a unit of land the household owns l The household decides to produce or not after observing the yield shock. I assume the two stochastic processes are independent of each other. The problem of the household is;

$$\begin{aligned} V_{i,j}(l) &= \max_{c,l'}\{U(c)+\beta\sum_{i',j'}\Gamma\Lambda V_{i',j'}(l')\} \\ s.t.\ \ c+q_{ij}l' &= [\max\{0,d_i-\alpha\}+q_{ij}]l+z_j \\ q_{ij}l' &\leq [\max\{0,d_i-\alpha\}+q_{ij}]l \\ q_{ij} &= Q(d_i,z_j)\ \ given \end{aligned}$$

where q_i is the price of a share of land in state i

Question 7

A RCE is a list of functions $\{V(.),\, Q(.),\, g(.)\}$ such that,
-given $Q(.), \{V(.),\, g(.)\}$ solves the problem of the household
-equilibrium share of land of HH is 1

$$g(d,z,1)=1$$

-market clearing

$$c=z+\max\{0,d-\alpha\}$$

Question 8

The price of such option is the appropriately discounted value of positive consumption bundles on the two period event tree. Formally in AD notation,

$$q^o(p_1,p_2)=\sum_{h_t|h_{t+1}}\sum_{h_{t+1}|h_{t+2}}\max\{0,[q(h_{t+1})-p_1]p(h_{t+1})+[p_2-q(h_{t+2})]p(h_{t+2})\}\frac{1}{p(h_t)}$$

where $p(h_t)=\frac{\beta\pi(h_t)U_c(c(h_t))}{U_c(c(h_0))}$ is the usual Arrow-Debrue price of a unit of consumption good after history h_t and $q(h_t)$ is the price of land after history h_t given, that is $\sum_{s=0}^{\infty}p(h_{t+s})\max\{0,d(h_{t+s})-\alpha\}=\beta\sum_{s=0}^{\infty}\frac{\pi(h_{t+s})U_c(c(h_{t+s}))}{\pi(h_t)U_c(c(h_t))}\max\{0,d(h_{t+s})-\alpha\}$ in recursive form the price of land in state (d_i,z_j) is,

$$q_{ij}=\frac{\sum_{i'}\sum_{j'}\Gamma_{ii'}\Lambda_{jj'}U_c(\max\{0,d_{i'}-\alpha\}+z_{j'})[\max\{0,d_{i'}-\alpha\}+q_{j'}]}{U_c(\max\{0,d_i-\alpha\}+z_j)}$$

and the price of state contingent asset to deliver one unit of consumption in state $i'j'$ is,

$$p_{ij}^{i'j'}=\frac{\sum_{i'}\sum_{j'}\Gamma_{ii'}\Lambda_{jj'}U_c(\max\{0,d_{i'}-\alpha\}+z_{j'})}{U_c(\max\{0,d_i-\alpha\}+z_j)}$$

then in recursive language the price of such an option is state ij is,

$$q_{ij}^o(p_1,p_2)=\sum_{ij|i'j'}\sum_{i'j'|i''j''}\max\{0,\left([q_{i'j'}-p_1]+[p_2-q_{i'',j''}]p_{i'j'}^{i'',j''}\right)p_{ij}^{i',j'}\}$$

noting that the second term in non-zero term in max operator is the value of the option in period one and first term is the value of the gain or loss by buying the share at price p_1 at period one, both of which are denominated in period one ahead consumption good, to get the value of the option in terms of period zero consumption good it is multiplied by the appropriate state contingent prices in each node.

Question 9

The land is not used for production for two consequetive periods iff given this period's yield shock d_i,

$$(d_i - \alpha) + \sum_{i|j} \Lambda_{ij} \max\{0, d_j - \alpha\} \leq 0$$

we also know if the land is left idle for two periods given this periods endowment shock z_i,

$$c = z_i \quad , \quad c' = z_j \quad for \quad some \quad j$$

then from the household problem given this periods shocks we can get the familiar formulation for the price of land in first period of idleness as a function of state ij by solving the following system of linear equations;

$$q_{ij} = \beta \sum_{ij|i'j'} \frac{\Gamma_{ij}\Lambda_{ij}U'(z_j)}{U'(z_i)} q_{i'j'}$$