Suggested Solutions to Problem Set 5
Econ 702 Spring 2005

Solution 1

The Gini Index is given by the following formula:

$$\frac{\text{Area enclosed by the Lorenz Curve and 45 degree line}}{\text{Triangle made by connecting the points (0,0), (1,0), (1,1)}}$$

Define A=$[0,\overline{a}]$ (where a is wealth). Let our measure of agents be $\mu(a)$. Note that if we have a mapping defining lorenz curve, we can find the Gini index by integration.

Let $x \in B = [0,1]$ denote the percentage of population when our agents are ordered increasingly with respect to their wealth level and also let $y \in C = [0,1]$ denote the cumulative share of wealth.Then define f to be

$$f : B \to C$$

our Lorenz curve. To define f, as an intermediate step lets define the mapping $z : B \to A$ as,

$$\int_0^{z(x)} \mu(a)da = x \tag{1}$$

so given x, a cut-off point for our population, the mapping z gives the wealth level that seperates the bottom x percent of our population from top $1-x$ percent. Then we can define our lorenz curve as ,

$$f(x) = \frac{\int_0^{z(x)} a\mu(a)da}{\int_0^{\overline{a}} a\mu(a)da} \tag{2}$$

Then formula for the Gini index is,

$$g = 2 * [0.5 - \int_0^1 f(x)dx] \tag{3}$$

Kurtosis measures the "fatness" of the tails of a distribution and the formula is,

$$k = \frac{\mu_4}{\sigma^4}$$

where μ_4 is the 4^{th} central moment of the distribution and σ is the standart deviation of the distribution. Then the formula in our framework is,

$$k = \frac{\int (a-\overline{a})^4 \mu(a)da}{\left(\int (a-\overline{a})^2 \mu(a)da\right)^2}$$

where,

$$\overline{a} = \int a\mu(a)da$$

is the mean wealth in our economy.

Solution 2

$$Q\left(s,a,B\right)=1_{g(s,a)\in B_a}\left(\sum_{s'\in B_s}\Gamma_{ss'}\right)\quad (Q(e,a,B)=\sum_{e_j\in B}\Gamma_{ee'}1_{[\{e_j,g(e,a)\}\in B]})$$

We need to show that Q constructed as above satisfies the conditions of a transition function, which are:

1. $\forall B\in\mathcal{A},\ \ Q(\cdot,B):A\to\mathcal{R}$ is measurable.
2. $\forall(e,a)\in A=S\times A,\ \ Q(e,a,\cdot)$ is a probability measure.

   Note that the definition of measurability requires,

   $$\{(e,a)\in A:Q(e,a,B)\leq c\}\in\mathcal{A}\text{ for all }c\in R$$

   To show (1), given $B\in\mathcal{A}$, B_a and B_e are fixed. Then $1_{g(s,a)\in B_a}$ $(1_{[\{e_j,g(e,a)\}\in B]})$has value 0 or 1 only. And since Γ is a transition matrix for a Markov chain, $\sum_{e'\in B_s}\Gamma_{ee'}$ is measurable by assumption. Therefore, $Q(\cdot,B)=1_{g(e,a)\in B_a}\cdot\sum_{e'\in B_s}\Gamma_{ee'}$ is measurable through measurability of products of measurable functions.

To show (2), given $(e,a)\in A=S\times A$, we need to check the following properties for probability measure

(a).The empty set has measure 0,

$$Q(s,a,\oslash)=0\tag{4}$$

That is $B=\oslash$, then $B_a=\oslash$and $B_s=\oslash$. Therefore $1_{g(s,a)\in B_a}=0$ and $\sum_{s'\in B_s}\Gamma_{ss'}=0$. Done.

(b).Always non-negative for any event and normalized to one for the whole set of events,

$$\begin{aligned}Q(s,a,B)&\geq 0\text{ for all }B\in\mathcal{A}&(5)\\ Q(s,a,S\times A_a)&=1&(6)\end{aligned}$$

Non-negativity is trivial since $1_{[\{e_j,g(e,a)\}\in B]}$ takes only values of 0 or 1 and the entries of Markov matrix is non-negative by definition. Normalizing to unity for $B=S\times A$, then $B_a=A$ and $B_s=S$, follows from the agent's decision rule $g(e,a)\in[\underline{a},\overline{a}]=A$ is in the asset space A for some finete $[\underline{a},\overline{a}]$ to be determined endogenously as we will see soon, thus $1_{g(s,a)\in A}=1$. And by definition the property of Markov transition matrix, $\sum_{e'\in S}\Gamma_{ee'}=1$. Therefore $Q(e,a,S\times A)=1$

(c).Measure of union of countable infinite set of disjoint subsets of $\mathcal{A}$ are equal to the sum of their measures. For the case of two subsets the following has to hold and this can be generalized to countable infinity,

$$Q(e,a,B_i\cup B_j)=Q(e,a,B_i)+Q(e,a,B_j)\text{ if }B_i\cap B_j=\oslash\tag{7}$$

Case 1): If $g(s,a) \notin (B_i \cup B_j)_a$, then $g(s,a) \notin B_{i,a}$ and $g(s,a) \notin B_{j,a}$. So $1_{g(s,a) \in B_a} = 0$ for all $(B_i \cup B_j)_a$, $B_{i,a}$ and $B_{j,a}$. Thus (**??**) holds trivial.

Case 2): If $g(e,a) \in (B_i \cup B_j)_a$, $1_{g(s,a) \in (B_i \cup B_j)_a} = 1$. Note that $g(e,a) \in (B_i)_a$ or $g(e,a) \in (B_j)_a$ since they are disjoint and g is a well defined function, w.l.o.g. assuming $g(e,a) \in (B_i)_a$then we know $Q(e,a,B_i \cup B_j) = Q(e,a,B_i \cup (B_j)_s)$ and,

$$\begin{aligned} 1_{g(s,a) \in (B_i \cup B_j)_a} &= 1_{g(s,a) \in (B_i)_a} + 1_{g(s,a) \in (B_j)_a} \\ Q(e,a,B_i \cup B_j) &= \sum_{e' \in (B_i \cup B_j)_s} \Gamma_{ee'} 1_{[\{g(e,a)\} \in (B_i \cup B_j)_a]} = \\ & \sum_{e' \in (B_i \cup B_j)_s} \Gamma_{ee'} 1_{[\{g(e,a)\} \in (B_i)_a]} + \sum_{e' \in (B_i \cup B_j)_s} \Gamma_{ee'} 1_{[\{g(e,a)\} \in (B_j)_a]} \end{aligned}$$

and one of the summation terms in RHS has to be equal to zero. As assumed the second term is zero then,

$$Q(e,a,B_i \cup B_j) = \sum_{e' \in (B_i \cup B_j)_s} \Gamma_{ee'} 1_{[\{g(e,a)\} \in (B_i)_a]}$$

and we also know $(B_i \cup B_j)_s = \oslash$ and by the property of the Markov transition matrix,

$$\sum_{e' \in (B_i \cup B_j)_s} \Gamma_{ee'} 1_{[\{g(e,a)\} \in (B_i)_a]} = \sum_{e' \in (B_i)_s} \Gamma_{ee'} 1_{[\{g(e,a)\} \in (B_i)_a]} + \sum_{e' \in (B_j)_s} \Gamma_{ee'} 1_{[\{g(e,a)\} \in (B_i)_a]}$$

and thus

$$Q(e,a,B_i \cup B_j) = Q(e,a,B_i \cup (B_j)_s) = Q(e,a,B_i) + Q(e,a,B_j)$$

and Q is indeed a transition function. Note that throughout the solution it is assumed $B \subset S \times A$ i.e. B is representable as a cross product.

Solution 3

$$\Gamma = \begin{bmatrix} \Gamma_{ee} & \Gamma_{eu} \\ \Gamma_{ue} & \Gamma_{uu} \end{bmatrix}$$

Stationary distribution is eigenvector associated with eigen value 1.

$$\Gamma^T x^* = x^*$$

where $x^* = (e^*, u^*)$

$$\begin{aligned} \Gamma_{ee} e + \Gamma_{ue} u &= e \\ \Gamma_{eu} e + \Gamma_{uu} u &= u \end{aligned}$$

And we normalized

$$e + u = 1$$

Thus, the stationary distribution for employment states is,

$$x^* = \left[\frac{\Gamma_{ue}}{1 - (\Gamma_{ee} - \Gamma_{ue})}, \frac{1 - \Gamma_{ue}}{1 - (\Gamma_{ee} - \Gamma_{ue})} \right]$$

Solution 4

The problem here is that the state space for an agent is not compact. For the neoclassical growth model, we had a production technology and we made the assumption of Inada conditions on the production function. This guarantees that the level of capital stock in equilibrium stays bounded because of the curvature in the production function, saving too small is not optimal since the marginal productivity of capital goes up without bound and saving too much is not optimal either because the marginal productivity of capital goes to zero. However in this problem, there is nothing that is preventing our fishermen from storing too many fish. In other words there is no upper bound for a'. Note that there is a lower bound which is imposed by Mother Nature: You can't store negative amounts.

We use the following theorem to show that there is an upper bound:

Theorem 5 *If $\beta < \frac{1}{1+r}$ $=q$, then $\exists \overline{a}$ such that, if $a_0 < \overline{a}$, $a_t < \overline{a} \quad \forall t$.*

If $\beta < \frac{1}{1+r}$, this means that you are too impatient and even the returns that you will get from saving today, r, will not offset that impatience.You'd rather consume today than later (you're discounting the future too much to wait). So if you are impatient enough compared with the returns from technology, gains from saving disappear eventually, and you stop saving more. And hence we have an upper bound on savings.

Given the compact domain, we can apply the Contraction Mapping Theorem if this problem can be represented as an operator that is a contraction and the solution exists and it is unique.

Theorem 6 *(Contraction Mapping Theorem) If ($\mathcal{M}$,d) a complete metric space and T: M→ M is a contraction then,*

1. *T has a unique fixed point.*
2. *The sequence $\{\varphi_n\}$ such that $\varphi_{n+1} = T\varphi_n$ will converge to V for any initial condition V_0.*

You can easily show with the appropriate assumptions this operator is indeed a CM using Blackwell's sufficient conditions and your skills obtained in 704.