Econ 702, Spring 2005
Problem Set 7
Suggested Answers

Problem 1 Hugget Economy

Will be added very soon.

Growth Models

Problem 2

First of all, it should be clear that in this context there is no uncertainty. Recall from the general case (with uncertainty) that the price of fruit is given by

$$p_t(h_t) = \beta^t \pi(h_t) \frac{u_c(d_t)}{u_c(d_0)} = \beta^t \pi(h_t) \left(\frac{d_t}{d_0}\right)^{-\sigma},$$

which in the present case will become

$$\beta^t \left(\frac{d_t}{d_0}\right)^{-\sigma} = \beta^t \left(\frac{\gamma^t d_0}{d_0}\right)^{-\sigma} = \left(\beta\gamma^{-\sigma}\right)^t$$

Now the price of a tree whose fruit grows at rate γ per period will be given by:

$$P(\gamma) = \sum_{t=0}^{\infty} p_t \ d_t = \sum_{t=0}^{\infty} \left(\beta\gamma^{-\sigma}\right)^t \ d_t = \sum_{t=0}^{\infty} \left(\beta\gamma^{-\sigma}\right)^t \ \gamma^t d_0 = d_0 \sum_{t=0}^{\infty} \left(\beta\gamma^{1-\sigma}\right)^t.$$

So the value of having the higher growth rate can be computed as:

$$P(0.02) - P(0.01) = d_0 \sum_{t=0}^{\infty} \left(\beta (0.02)^{1-\sigma}\right)^t - d_0 \sum_{t=0}^{\infty} \left(\beta (0.01)^{1-\sigma}\right)^t =$$

$$= d_0 + \sum_{t=1}^{\infty} \left(\beta \left[(0.01)^{1-\sigma} - (0.01)^{1-\sigma}\right]\right)^t$$

Problem 3

The statement is wrong. consider the function

$$F(k,l) = \tau \left[ak^{-\rho} + (1-a)\,l^{\ -\rho}\right]^{-1/\rho}$$

Let's work with labor. The same results hold (by symmetry) for the other factor. Under perfect competition the wage is

$$w = \frac{\partial F}{\partial l} = \tau \left[ak^{-\rho} + (1-a)\,l^{\ -\rho}\right]^{-1/\rho - 1} (1-a)\,l^{\ -\rho-1}$$

and the labor's payment is

$$\tau \left[ak^{-\rho} + (1-a)\,l^{\ -\rho}\right]^{-1/\rho - 1} (1-a)\,l^{\ -\rho}$$

So the share of labor payments over total output is

$$s_l = \frac{\tau \left[ak^{-\rho} + (1-a)\,l^{\ -\rho}\right]^{-1/\rho - 1} (1-a)\,l^{\ -\rho}}{\tau \left[ak^{-\rho} + (1-a)\,l^{\ -\rho}\right]^{-1/\rho}}$$

or after a little algebra

$$s_l = (1-a)\left[a\,(k/l)^{-\rho} + 1 - a\right]^{-1}$$

So s_l clearly changes as the capital-labor ratio changes. The claim is not true.

Problem 4 Consider the following problem of the Social Planner (in an economy with constant population growth γ)

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ N_t \ u\left(\frac{c_t}{N_t}\right)$$

$$s.t: \quad c_t + k_{t+1} = F(k_t, N_t) + (1-\delta)\,k_t,$$

where $N_t = \gamma^t N_0$.

For any variable define $\hat{x}_t = \frac{x_t}{\gamma^t}$. Then the objective function can be rewritten as (we also normalize $N_0 = 1$):

$$\max_{c_t, k_{t+1}} \sum_{t=0}^{\infty} (\beta\gamma)^t \quad u\left(\hat{c}_t\right) \text{ and the constraint as}$$

$$\hat{c}_t + \gamma \hat{k}_{t+1} = F\left(\hat{k}_t, 1\right) + (1-\delta)\,\hat{k}_t, \text{ and define } f(x) = F(x,1).$$

The recursive formulation of the problem is the following:

$$V(\hat{k}) = \max_{\hat{k}'} \left[u\left(f\left(\hat{k}\right) + (1-\delta)\,\hat{k} - \gamma \hat{k}' \right) + \beta\gamma V(\hat{k}') \right]$$

First Order Condition:

$$u_c\left(\hat{c}\right)(-\gamma) + \beta\gamma V'(\hat{k}') = 0$$

and envelope condition is $V'(\hat{k}) = u_c\left(\hat{c}\right)\left[f'\left(\hat{k}\right) + 1 - \delta\right]$, yielding

$$\gamma u_c\left(\hat{c}\right) = \beta\gamma u_c\left(\hat{c}'\right)\left[f'\left(\hat{k}'\right) + 1 - \delta\right]$$

At Steady State, and with a CD production function we have

$$\hat{k}^* = \left(\frac{1-\beta+\beta\delta}{a\beta A}\right)^{1/(a-1)}$$

Since the hat economy is at S.S $\hat{k}, \hat{c}$ are constant, i.e,

$\frac{k_t}{\gamma^t} = m$ (where m is a constant), so $k_t = m\gamma^t$ and thus, $\frac{k_{t+1}}{k_t} = \gamma$, which

means that the original economy is on a Balanced Growth Path.

Problem 5

In the Social Planner's problem the objective function was such that it was easy to apply the hat transformation. This will not be true in the competitive setting. In order to overcome this technical difficulty we will work with an economy of representative households (as opposed to agents). Each of the households grows at a constant rate γ, and so does the economy as a whole.

The objective of a representative household is to

$$\max_{c_t,k_{t+1}} \sum_{t=0}^{\infty} \beta^t \ n_t \ u\left(\frac{c_t}{n_t}\right), \text{ where } n_t \text{ is the number of members of the household at } t,$$

$$s.t: \quad c_t + a_{t+1} = \ a_t\,(1 + r_t - \delta) + n_t w_t,$$

$$\text{where } n_t = \gamma^t n_0 = \gamma^t.$$

Applying the same hat transformation we have

$$\max_{c_t,k_{t+1}} \sum_{t=0}^{\infty} (\beta\gamma)^t \quad u\left(\hat{c_t}\right),$$

$$s.t: \quad \hat{c_t} + \gamma \hat{a}_{t+1} = \ \hat{a_t}\,(1 + r_t - \delta) + w_t,$$

The recursive problem is

$$V\left(\hat{a}, \hat{k}\right) = \max_{\hat{a}'} \left[u\left((1 + r - \delta)\,\hat{a} + \hat{w} - \gamma \hat{a}'\right) + \beta\gamma V\left(\hat{a}', \hat{k}'\right)\right]$$

$$s.t: \ r = r\left(\hat{k}\right), \quad w = w\left(\hat{k}\right), \ \hat{k}' = G\left(\hat{k}\right)$$

It's easy to derive the FOC and Envelope Condition and verify that the solution to this problem coincides wit that of the Social Planner.

Finally the definition of a Recursive Competitive Equilibrium is as follows:

DEFINITION: A RCE is a list of functions $\{V\left(\right),\ g\left(\right), G\left(\right),\ r\left(\right),\ w\left(\right)\}$, such that:

1) Agents (households) maximize: given $G\left(\right),\ r\left(\right),\ w\left(\right)$, the functions $V\left(\right),\ g\left(\right)$, solve the household's problem.

2) Firms maximize: $r(),\ w()$ solve the firm's problem.

3)Representative household condition: $g\left(\hat{k},\hat{k}\right)=G\left(\hat{k}\right)$.

Problem 6

We will use the transformation $\hat{c_t}=c_t/\gamma_n^t$, $\gamma_n>1$.

The problem becomes:

$$\max_{c_t,k_{t+1}} \frac{1}{1-\sigma}\sum_{t=0}^{\infty}\left(\beta\gamma^{1-\sigma}\right)^t \left(\hat{c_t}\right)^{1-\sigma}$$

$$\text{s.t: } \hat{c_t}+\gamma_n\hat{k}_{t+1}=f\left(\hat{k_t}\right)+(1-\delta)\,\hat{k_t}$$

We can also write this problem recursively as

$$V(\hat{k})=\max_{\hat{k}'}\left[u\left(f\left(\hat{k}\right)+(1-\delta)\,\hat{k}-\gamma_n\hat{k}'\right)+\beta\gamma^{1-\sigma}V(\hat{k}')\right]$$

First Order Condition:

$$u_c\left(\hat{c}\right)(-\gamma_n)+\beta\gamma^{1-\sigma}V'(\hat{k}')=0$$

and envelope condition is $V'(\hat{k})=u_c\left(\hat{c}\right)\left[f'\left(\hat{k}\right)+1-\delta\right]$, yielding

$$\gamma_n u_c\left(\hat{c}\right)=\beta\gamma_n^{1-\sigma}u_c\left(\hat{c}'\right)\left[f'\left(\hat{k}'\right)+1-\delta\right]$$

At Steady State with a CD function:

$\hat{k}^*=\left(\frac{1-(1-\delta)a\beta\gamma_n^{-\sigma}}{a\beta\gamma_n^{-\sigma}}\right)^{1/(a-1)}$, which depends on the parameter σ, because this parameter indicates how patient people are.

Moreover, the original economy will be on a BGP, where all variables grow at rate γ_n.

Problem 7

Throughout this exercise we assume that the production function is C.D.
Capital augmenting technological progress: $Y_t = F\left(\gamma_k^t\ k_t, N_t\right)$.
Note that we can write

$$F\left(\gamma_k^t\ k_t, N_t\right) = \left(\gamma_k^t k_t\right)^a N_t^{1-a} = k_t^a \left[\left(\left(\gamma_k\right)^{\frac{a}{1-a}}\right)^t N_t\right]^{1-a} = k_t^a \gamma^t N_t,$$

where we defined $\gamma = \left(\gamma_k\right)^{\frac{a}{1-a}}$.

But this is a problem of labor augmenting technological process, and we solved it in the previous exrcise. So we know that this economy will be on a BGP where all variables grow at rate $\left(\gamma_k\right)^{\frac{a}{1-a}}$.

Total productivity technological progress: $Y_t = \gamma^t F\left(\gamma_k^t\ k_t, \gamma_n^t\ N_t\right)$

Write

$$Y_t = \gamma^t F\left(\gamma_k^t\ k_t, \gamma_n^t\ N_t\right) = F\left(\bar{\gamma}_k^t\ k_t, \bar{\gamma}_n^t\ N_t\right), \text{ where } \bar{\gamma}_i^t = \gamma^t \gamma_i^t\ ,\ i = k, n.$$

Define $\hat{x}_t = \frac{x_t}{\bar{\gamma}_n^t}$. Then the objective is to

$$\max \sum \left(\beta \left(\gamma\gamma_n\right)^{1-\sigma}\right)^t \left(\hat{c}_t\right)^{1-\sigma}$$

$$\text{s.t: } \hat{c}_t + \bar{\gamma}_n \hat{k}_{t+1} = F\left(\frac{\bar{\gamma}_k^t}{\bar{\gamma}_n^t}\ k_t,,\ N_t\right) + (1-\delta)\,\hat{k}_t$$

Let $g^t = \frac{\bar{\gamma}_k^t}{\bar{\gamma}_n^t}$. Then the production function is

$$Y_t = \left(g^t k_t\right)^a N_t^{1-a} = k_t^a \left[\left(g^{\frac{a}{1-a}}\right)^t N_t\right]^{1-a} = k_t^a \gamma^t N_t, \text{ where } \gamma^t = \left(g^{\frac{a}{1-a}}\right)^t = \left(\left(\frac{\bar{\gamma}_n}{\bar{\gamma}_k}\right)^{a/(1-a)}\right)^t$$

But this is an economy with labor augmenting technological process. Hence we know that (after a few inverse applications of the above transformations) the original economy will grow at rate

$$\gamma\left(\gamma_k\right)^{a/(1-a)}\left(\gamma_n\right)^{(1-2a)/1-a}$$

Investment specific technological progress: $k_{t+1} = \gamma_i^t \; i_t + (1-\delta)\,k_t$. We have:

$$k_{t+1} = \gamma_i^t \left[\, F\left(k_t, N_t\right) - c_t\right] + (1-\delta)\,k_t$$

$$\text{or defining } \hat{x}_t = \frac{x_t}{\gamma_i^t}$$

$$\gamma_i \; \hat{k}_{t+1} = k_t^a N_t^{1-a} - c_t + (1-\delta)\,\hat{k}_t \quad \textit{or}$$

$$c_t + \gamma_i \; \hat{k}_{t+1} = \left(\hat{k}_t^{\,a}\right) \left[\left(\left(\gamma_i\right)^{a/(1-a)}\right)^t N_t\right]^{1-a} + (1-\delta)\,\hat{k}_t$$

which is a problem of labor augmenting technological process.

Note, however, that here the growth rate of consumption and capital are not going to be the same, because in the xepression above we have "hats" only for capital and not for consumption. We have:

$$g_c = \left(\gamma_i\right)^{a/(1-a)} \quad \textit{and} \quad g_k = \gamma_i \; \left(\gamma_i\right)^{a/(1-a)} = \left(\gamma_i\right)^{1/(1-a)},$$

and clearly $g_k > g_c$.