Problem Set 8
Suggested Solutions

Solution 1

Note that before we go any further feasibility of any sort of growth requires the net return on capital (linear and constant in this case) is greater than unity on the BGP. Then a necessary condition is,

$$\begin{aligned}(1-\delta)k + Ak &= (1-\delta+A)k \\ \widetilde{A} &= (1-\delta+A) > 1 \\ A &> \delta\end{aligned}$$

we also know with the usual CRRA preferences, the growth rate on BGP is,

$$\gamma = (\beta A)^{\frac{1}{\sigma}}$$

then growth means,

$$\begin{aligned}\gamma &> 1 \\ (\beta A)^{\frac{1}{\sigma}} &> 1\end{aligned}$$

We are not done yet. We also have to make sure the transversality condition is satisfied, i.e. the total utility on the optimal solution path is bounded, otherwise we do not have a solution. Writing the total utility on BGP,

$$\sum_t \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

then substituting $c_t = c_0 * \gamma^t$ one can get,

$$\sum_t [\beta(\beta A)^{\frac{1-\sigma}{\sigma}}]^t \frac{c_0^{1-\sigma}}{1-\sigma}$$

then the last necessary condition is the term in square brackets is less than one,

$$\beta(\beta A)^{\frac{1-\sigma}{\sigma}} < 1.$$

Solution 2

The technology is given by,

$$Y_t = F(K_t, A_t N_t) \quad (1)$$

$$A_{t+1} = \gamma * A_t \quad (2)$$

The problem is,

$$\max \sum_{t=0}^{\infty} \beta^t U(C_t) \tag{3}$$
$$st \quad C_t + K_{t+1} = F(K_t, \gamma^t A_0 N_t) + (1-\delta)K_t$$

Our tools allow us to solve only stationary problems yet this problem is nonstationary due to 'labor augumenting' growth. We have to normalize the variables to 'per efficiency labor' units, by dividing all by γ^t and transform this economy to a stationary one. Also normalize $A_0 = 1$ and we also know since there is no leisure in the utility, $N_t = 1$ forall t.Then the problem becomes,

$$\max \sum_{t=0}^{\infty} \beta^t U(\gamma^t \widehat{c}_t) \tag{4}$$
$$st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} = F(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t$$

Suppose we have a CRRA preferences, then the question is how can we represent the preferences as a function of $\widehat{c}_t$ only. Writing the CRRA,

$$\sum_{t=0}^{\infty} \beta^t \frac{(\gamma^t \widehat{c}_t)^{1-\sigma}}{1-\sigma} = \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma}}{1-\sigma} \tag{5}$$

and the problem becomes,

$$\max \sum_{t=0}^{\infty} (\beta(\gamma^{1-\sigma}))^t \frac{\widehat{c}_t^{\,1-\sigma}}{1-\sigma} \tag{6}$$
$$st \quad \widehat{c}_t + \gamma \widehat{k}_{t+1} = F(\widehat{k}_t, 1) + (1-\delta)\widehat{k}_t$$

Under usual regularity conditions and provided that $\beta(\gamma^{1-\sigma}) < 1$ we know there exist an optimal solution to this problem with a stable steady state. We also know the SS of this economy corresponds to the BGP of the original economy. One can derive the EE of this economy at the SS which is,

$$\gamma^\sigma = \beta[(1-\delta) + F_1(\widehat{k}^*, 1)]$$

where $\widehat{k}^* = (\frac{k_t}{\gamma^t})^* = (\frac{k_t}{A_t})^*$. Remember our usual assumptions about the technology $F_{11} < 0$, that is our technology is concave in each of its arguments, then we know F_1 is a monotone function and given γ the growth rate of technological change, there exist a unique $(\frac{k_t}{A_t})^*$ that solves this equation. If the economy starts with any other ratio, it will follow a transition path towards this steady state, where it will exhibit variable growth rates (that is the output will have a variable growth rate), even though the labor efficiency grows at fixed rate γ. This is because the capital stock grows at a variable rate.

Remembering the Euler equation of the AK model with BGP assumption,

$$\gamma^\sigma = \beta[(1-\delta) + A]$$

we see that the state variable capital stock drops out of the equation. This implies no transitional dynamics and complete history dependence in the sense where you start basically determines your fate. Any initial capital level will put you on a BGP with the same growth rate where only the levels differ. So if two country starts with same set of technological and preference parameters and only differ in initial capital, the difference persist forever and there is no chance of cross country convergence, which is in contrast with some observed growth facts.

Solution 3

We know in our dynamic optimization problems solution requires equating MRS over time. We also know that the defining character of a BGP is the constant growth rates of variables, that is the ratio, of these variables between periods is constant $((c_{t+1}/c_t) = \gamma)$. CRRA preferences have the nice property that MRS is only a function of ratios rather than levels of the variables thus on a BGP, solution is not violated (i.e. if a BGP exists and once you are on it is consistent with optimizing behaviour in the sense consumption ratios are constant over the solution path and thus MRS), that is CRRA preferences are compatible with a BGP. It turns out these are the only class of preferences that has this property of being compatible with a BGP but showing necessity requires a little bit of differential equation skills and will not be pursued here.

Solution 4

We know from the class notes that one of the FOC conditions that characterize solution is,

$$(\frac{c_{t+1}}{c_t})^{\sigma} = \beta A[K_{t+1}^{\alpha-1} H_{t+1}^{1-\alpha} + (1-\delta^k)]$$

imposing the balanced growth conditions on this equation will give us,

$$\gamma_c^{\sigma} = (\frac{\gamma_k}{\gamma_h})^{(t+1)(\alpha-1)} \beta A (\frac{K_0}{H_0})^{\alpha-1} + (1-\delta^k)$$

where 'gamma' subscripts refer to the growth rate of the relevant variable. First thing that we are asked to show is the growth rates for human capital and physical capital are equal on a BGP. It is easy to establish that since if it is not the case we know $\gamma_c^{\sigma} \rightarrow \infty$ if $(\gamma_k > \gamma_h)$ or $\gamma_c^{\sigma} \rightarrow (1-\delta^k)$ if $(\gamma_k < \gamma_h)$ and the growth rates of human and physical capital are not constant.

Next as an intermediate step, lets show if $\gamma_k = \gamma_h$ then $\frac{X_{t+1}^h}{X_t^h} = \frac{X_{t+1}^k}{X_t^h}$. Writing the law of motion for human capital and imposing BGP conditions,

$$\begin{aligned} X_t^h &= H_{t+1} - H_t(1-\delta^h) \\ &= \gamma_h^t H_0[\gamma_h - (1-\delta)] \end{aligned}$$

then writing the same expression for X_{t+1}^h and dividing them,

$$\frac{X_{t+1}^h}{X_t^h} = \frac{\gamma_h^{t+1}}{\gamma_h^t} \frac{H_0[\gamma_h - (1-\delta)]}{H_0[\gamma_h - (1-\delta)]} = \gamma_h$$

going through the same steps for physical capital will give γ_k as the growth rate of investment in physical capital and we are done with this step. With these in hand we know our technology is CRTS and both inputs are growing at a rate $\gamma_k = \gamma_h = \gamma$ thus $\frac{Y_{t+1}}{Y_t} = \gamma$. Then from feasibility,

$$C_{t+1} = Y_t - X_t^h - X_t^k = \gamma^{t+1}[Y_0 - X_0^h - X_0^k]$$

and $\frac{C_{t+1}}{C_t} = \gamma$. Thus we are done showing,

$$\gamma_c = \gamma_k = \gamma_h = \gamma$$

and γ and $(\frac{K_0}{H_0})$ can be pinned down using the FOC corresponding to the human capital with the FOC above.

Solution 5

The question is not very explicit about how leisure enters the production funciton and preferences or the law of motion for the human capital. You could proceed with different assumptions as long as you are consistent. Below, I will setup a model of 'on the job training' i.e. the time spent in production will be contributing to the accumulation of human capital. Another way of formulating the model would involve a decision about how to allocate the time that is not consumed as leisure between production and human capital accumulation. Of course anything in between these two are possible as well. The choice depends on the question that you are trying the answer using the model. Consider,

$$\max \sum \frac{c_t^{1-\sigma}(vl_t) - 1}{1-\sigma}$$

$$\begin{aligned} s.t. \;\; k_{t+1} + c_t &= Ak_t^\alpha[h_t(1-l_t)]^{1-\alpha} + (1-\delta^k)k_t \\ h_{t+1} &= h_t(1-\delta^h) + (1-l_t)\zeta \end{aligned}$$

Note that the time spent in production $(1-l_t)$ linearly contributes to the accumulation of human capital. As mentioned in class, the reason for the impossibility of a BGP in this economy is due to the natural bound on the human capital accumulation. Even if the agent works all the time $l_t = 0$ the maximum possible human capital that can be accumulated is $\overline{h} = \frac{\varsigma}{\delta^h}$. Once this bound is hit, accumulating physical capital will run into diminishing returns and growth rate stalls. Deriving the SS of this economy is an straightforward exercise.

Solution 6

Consider a tax on capital income which are distributed back as lump-sum transfers. The budget constraint of HH is,

$$T_t + c_t + k_{t+1} = r_t(1 - \tau_t)k_t + (1 - \delta)k_t$$

with a CRRA preference the EE is,

$$(\frac{c_{t+1}}{c_t})^{\sigma} = \beta[r_{t+1}(1 - \tau_{t+1}) + (1 - \delta)]$$

we know under competition $r_{t+1} = F_k(.) = \alpha A$ from our class notes. Then on a BGP,

$$\gamma_{ce}^{\sigma} = \beta[\alpha A(1 - \tau_{t+1}) + (1 - \delta)]$$

we also derived the BGP Euler equation for the social planner which was,

$$\gamma_{sp}^{\sigma} = \beta[A + (1 - \delta)]$$

then the exercise here is to pin down the tax rates that equates competitive growth rate to the optimal one which gives,

$$\tau_{t+1} = \tau = 1 - \frac{1}{\alpha} \text{ forall } t$$

that is a constant proportional subsidy ($\alpha < 1$) on capital income to induce households save at a higher rate which is financed by lumpsum taxes given by,

$$T_t = \tau \alpha A k_t$$