Econ 702, Spring 2005
Problem Set 9
Suggested Answers

Solution 1 *(Romer Endogenous Growth) Consider the model of Romer we covered in class with production composed of three sectors, final good, intermediate goods, R&D with final and intermediate good technology and law of motion for variety of new intermediate goods,*

$$\begin{aligned} Y_t &= L_{1t}^{\alpha} \int_0^{A_t} x_t(i)^{1-\alpha} di \\ (1-\delta)K_{t-1} + i_t &= K_t = \int_0^{A_t} \eta x_t(i) di \quad , K_0 \textit{ given} \\ A_{t+1} &= A_t + L_{2t}\zeta A_t \ , \ \ A_0 \textit{ given} \\ L_{1t} + L_{2t} &= 1 \end{aligned}$$

Assume CRRA preferences, formulate and solve the SPP's problem

Utilizing the fact that the curvature of the production function will imply $x_t(i) = x_t$ for all t, and assuming full depriciation $\delta = 1$,the SPP becomes,

$$\max_{\{C_t, K_{t+1}, A_{t+1}, L_{1t}, L_{2t}\}} \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma} - 1}{1-\sigma}\right) \tag{1}$$

$$\begin{aligned} s.t. \quad Y_t &= \frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}} A_t^{\alpha} K_t^{1-\alpha} \\ A_{t+1} &= (1 + L_{2t}\zeta)A_t \\ L_{2t} + L_{1t} &= 1 \\ Y_t &= C_t + K_{t+1} \end{aligned}$$

Before we proceed, couple of observations will be useful. First, looking at the reduced form production function, we see that in the eyes of the SP this is no different than 'labor augumenting' technological change, where the growth rate of the economy is determined by the growth rate of labor productivity, A_{t+1}/A_t. We also know, for such an economy, on the balanced growth path, $\gamma = A_{t+1}/A_t = K_{t+1}/K_t = Y_{t+1}/Y_t = C_{t+1}/C_t$and constant. Then $A_{t+1} = (1 + L_{2t}\zeta)A_t$ means $L_{2t} = L_{20}$ for all t (assuming economy starts on a BGP). Centralized problem here is to efficiently allocate the labor between two sectors and determine the optimal size of the R&D sector i.e. the number of varieties, so that the size of the R&D sector in turn will determine the growth rate of this economy on BGP. First note that,

$$A_{t+1} = \gamma A_t \Rightarrow \gamma = (1 + (1 - L_{10})\zeta) \tag{2}$$

then substituting consumption from feasibility constraint and production function into objective function and also substituting for the L_{2t} in the law of motion for A_t, we get he following Lagrangian,

$$L = \sum_{t=0}^{\infty}\left[\beta^t(\frac{(\frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}}A_t^{\alpha}K_t^{1-\alpha} - K_{t+1})^{1-\sigma}-1}{1-\sigma}) + \lambda_t(A_{t+1} - (1+(1-L_{1t})\zeta)A_t)\right] \tag{3}$$

The FOCs wrt. A_t and L_{1t},

$$(A_t) \quad : \quad \beta^t c_t^{-\sigma}[\frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}}\left(\frac{K_t}{A_t}\right)^{1-\alpha}\alpha] = -\lambda_{t-1} + \lambda_t(1+(1-L_{1t})\zeta) \tag{4}$$

$$(L_{1t}) \quad : \quad \beta^t c_t^{-\sigma}[\frac{L_{1t}^{\alpha-1}}{\eta^{1-\alpha}}\left(\frac{K_t}{A_t}\right)^{1-\alpha}\alpha](\frac{1}{\zeta}) = -\lambda_t \tag{5}$$

$$(K_{t+1}) \quad : \quad c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}[\frac{L_{1t}^{\alpha}}{\eta^{1-\alpha}}\left(\frac{K_t}{A_t}\right)^{-\alpha}(1-\alpha)] \tag{6}$$

(4) and (5) together imply,

$$L_{1t}\zeta = \frac{\lambda_{t-1}}{\lambda_t} - (1+(1-L_{1t})\zeta)$$

using the FOC for L_{1t} and constant growth rates on BGP $\frac{A_{t+1}}{A_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \gamma$ and $L_{1t} = L_{10}$ for all,

$$\beta\gamma^{-\sigma} = \frac{\lambda_t}{\lambda_{t-1}}$$

then,

$$L_{10}\zeta = (\gamma^{\sigma}\beta^{-1} - (1+(1-L_{10})\zeta)) \tag{7}$$

we also know,

$$\gamma = [(1+\zeta)\beta]^{1/\sigma} \tag{8}$$

Using the EE on a BGP and equation (2),

$$\begin{aligned}\gamma^{\sigma} &= \beta[\frac{L_{10}^{\alpha}}{\eta^{1-\alpha}}\left(\frac{K_0}{A_0}\right)^{-\alpha}(1-\alpha)] \\ \gamma &= (1+(1-L_{10})\zeta)\end{aligned}$$

we can solve for $\frac{K_0}{A_0}$ and L_{10}

Solution 2 *(Decentralized solution to Romer Endogenous growht model)*

From the FOC of the firm in the final goods sector, we have:

$$q_t(i) = (1-\alpha)L_{1t}^{\alpha}x_t(i)^{-\alpha} \tag{9}$$

$$w_t = \alpha L_{1t}^{\alpha-1}A_t x_t(i)^{1-\alpha} \tag{10}$$

and since $x_t = \frac{K_t}{\eta A_t}$, (10) becomes,

$$w_t = \alpha L_{1t}^{\alpha-1} A_t (\frac{K_t}{\eta A_t})^{1-\alpha} \tag{11}$$

From the FOC of the firm in the intermediate good sector:

$$(1-\alpha)^2 L_{1t}^{\alpha} (\frac{K_t}{\eta A_t})^{-\alpha} = R_t \eta \tag{12}$$

From the FOC of the firm in the R&D sector:

$$p_t^P = \frac{w_t}{\zeta A_t} \tag{13}$$

Also, in equilibrium, total profit a patent generates will be equal to the price of it so that the zero profit condition for the intermediate goods sector is satisfied.

$$p_t^P = \sum_{\tau=t}^{\infty} \frac{\pi_t(i)}{(R_t)^{\tau-t}} \tag{14}$$

Now write down the consumer's problem ($\delta = 1$):

$$\max_{c_t, L_{1t}, L_{2t}, k_{t+1}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1-\sigma} \right) \tag{15}$$

$$\text{s.t.} \qquad c_t + k_{t+1} = R_t k_t + w_t (L_{1t} + L_{2t})$$

From the FOC to the consumer's problem, we get:

$$(\frac{c_{t+1}}{c_t})^{\sigma} = \beta R_t \tag{16}$$

On the balanced growth path:

$$\begin{aligned} K_{t+1} &= \gamma K_t \quad A_{t+1} = \gamma A_t \quad c_{t+1} = \gamma c_t \\ L_{1t} &= L_1 \quad L_{2t} = L_2 \\ w_{t+1} &= \gamma w_t \quad x_t = x = \frac{K_t}{\eta A_t} \\ R_t &= R \quad \pi_t(i) = \pi \quad p_t^P = p^P \quad q_t(i) = q \end{aligned}$$

Use these above BGP conditions to write,
From (16),

$$\gamma^{\sigma} = \beta R \tag{17}$$

From (12),

$$(1-\alpha)^2 L_1^{\alpha} (\frac{K_t}{\eta A_t})^{-\alpha} = R\eta \tag{18}$$

From (13) and (11),

$$p^P = \alpha L_1^{\alpha-1} (\frac{K_t}{\eta A_t})^{1-\alpha} \frac{1}{\zeta} \tag{19}$$

Derive the expression for $\pi^*(i)$ by substituting for $x_t(i)$ from the FOC,

$$\begin{aligned}
\pi^*(i) &= (R\eta)^{\frac{\alpha-1}{\alpha}}\left[(1-\alpha)L_1((1-\alpha)^2L_1^{\alpha})^{\frac{\alpha}{1-\alpha}} - ((1-\alpha)^2L_1^{\alpha})^{\alpha}\right] \\
&= (R\eta)^{\frac{\alpha-1}{\alpha}}\phi(L_1) \qquad (20)
\end{aligned}$$

Then, rewrite zero profit condition (14) substituting for $\pi^*(i)$ using (20),

$$\begin{aligned}
p^P = \sum_{\tau=t}^{\infty}\frac{\pi_t(i)}{(R)^{\tau-t}} &= \frac{R}{R-1}\pi(i) \\
&= \frac{R^{\frac{2\alpha-1}{\alpha}}}{R-1}\eta^{\frac{\alpha-1}{\alpha}}\phi(L_1) \qquad (21)
\end{aligned}$$

Also from the law of motion for A,

$$A_{t+1} = \gamma A_t \text{ and } A_{t+1} = A_t + L_{2t}\zeta A_t \Rightarrow \gamma = (1 + L_2\zeta) \qquad (22)$$

Use (17),(18),(19),(21) and (22) to get the following system of equations,

$$\begin{aligned}
\gamma^{\sigma} &= \beta\eta^{\alpha-1}(1-\alpha)^2L_1^{\alpha}(\frac{K_0}{A_0})^{-\alpha} \qquad (23) \\
\gamma &= (1 + (1-L_1)\zeta) \qquad (24) \\
p^P &= \alpha\eta^{\alpha-1}L_1^{\alpha-1}(\frac{K_0}{A_0})^{1-\alpha}\frac{1}{\zeta} \qquad (25) \\
p^P &= \frac{(\eta^{\alpha-1}(1-\alpha)^2L_1^{\alpha}(\frac{K_0}{A_0})^{-\alpha})^{\frac{2\alpha-1}{\alpha}}}{(\eta^{\alpha-1}(1-\alpha)^2L_1^{\alpha}(\frac{K_0}{A_0})^{-\alpha}) - 1}\eta^{\frac{\alpha-1}{\alpha}}\phi(L_1) \qquad (26)
\end{aligned}$$

The above system of 4 equations (23)-(26) characterize $(\gamma, \frac{K_0}{A_0}, L_1, p^P)$.

Solution 3 *When a is not observable, planner can only choose c and V^u. And households choose a optimally. If given c and V^u, the agent will solve*

$$\max_a u(c) - a + \beta\left[p(a)V^E + (1-p(a))V^u\right] \qquad (27)$$

FOC is

$$[p'(a)\beta]^{-1} = V^E - V^u \qquad (28)$$

This FOC gives an implicit function of a as a function of V^u: $a = g(V^u)$.

Then, the planner solves her cost minimization problem,

$$c(V) = \min_{c,a,V^u} c + [1-p(a)]\beta c(V^u)$$

subject to

$$\begin{aligned}
V &= u(c) - a + \beta\left[p(a)V^E + (1-p(a))V^u\right] \qquad (29) \\
1 &= [p'(a)\beta]\left[V^E - V^u\right] \qquad (30)
\end{aligned}$$

Lagragian is

$$c+[1-p(a)]\beta c(V^u)+\theta\left[V-u(c)+a-\beta\left[p(a)V^E+(1-p(a))V^u\right]\right]$$
$$+\eta\left[1-[p'(a)\beta]\left[V^E-V^u\right]\right]$$

FOC: (c)

$$\theta^{-1}=u_c \tag{31}$$

(a)

$$c(V^u)=\theta\left[\frac{1}{\beta p'(a)}-\left(V^E-V^u\right)\right]-\eta\frac{p''(a)}{p'(a)}\left(V^E-V^u\right) \tag{32}$$

(V^u)

$$c'(V^u)=\theta-\eta\frac{p'(a)}{1-p(a)} \tag{33}$$

Envelope condition

$$c'(V)=\theta \tag{34}$$

We see the Lagrangian multiplier associated η is positive as long as the insurance is costly otw if $\eta=0$ (32) would imply $c(V^u)=0$ ($a>0\Rightarrow\frac{1}{\beta p'(a)}=\left(V^E-V^u\right)$), which means that the constraint is binding. So,

$$\eta\frac{p'(a)}{1-p(a)}>0$$

Therefore, we have

$$c'(V^u)<c'(V)\Rightarrow V^u<V$$

from the strict convexity of $c(.)$. The delayed promised utility decreases over time.

Let $\theta^u=\theta-\eta\frac{p'(a)}{1-p(a)}$, then $\theta^u<\theta$, which also implies θ is decreasing over time and thus so is consumption since

$$\theta^{-1}=u_c$$

And from

$$[p'(a)\beta]^{-1}=V^E-V^u$$

we know the effort level is decreasing over time. The reason is that V^u decreases over time, thus RHS increase over time and $p(.)$ is strictly concave, therefore, a_t is increasing over time.

Solution 4 *The production function is*

$$Y_t={N_t}^{1-a}\int_0^{A_t}x_t(i)^a\,di$$

To compute the elasticity of substitution between the inputs $x_t(i)$ and $x_t(j)$, set the output equal to a constant and apply total differentiation with respect to $x_t(i)$ and $x_t(j)$.We have

$$N_t^{\,1-a} a\ x_t(i)^{a-1}\, dx_t(i) + N_t^{\,1-a} a\ x_t(j)^{a-1}\, dx_t(j) = 0$$

So the elasticity of substitution between these factors is:

$$\left| \frac{dx_t(i)/x_t(i)}{dx_t(j)/x_t(j)} \right| = \left| -\left(\frac{x_t(j)}{x_t(i)}\right)^a \right| = \left(\frac{x_t(j)}{x_t(i)}\right)^a$$

Solution 5 *The problem is:*

$$\max_{c_t,\ k_{t\ +1}} \sum_{t=0}^{\infty} \beta^t\ \frac{c_t^{\,1-\sigma}}{1-\sigma}$$

$$s.t:\ c_t +\ k_{t+1} = z_t\ k_t^{\,a},$$

where $k_{t+1} \in \{k^L,\ k^H\}$, $z_t \sim\ \Gamma_{z\ z'}$ and $z_t = \{z^1,\ ...,\ z^N\}$.

Write the problem recursively:

$$V(k,z) = \max_{\substack{c, \\ k' \in \{k^L, k^H\}}} \{u(c) + \beta E V(k', z')\ \}$$

or

$$V(k,z) = \max_{k' \in \{k^L, k^H\}} \{u(zk^a - k') + \beta \Gamma_{zz'} V(k', z')\ \}$$

or

$$V(k,z) =$$
$$\max\ \left\{u\left(zk^a - k^L\right) + \beta \Gamma_{zz'} V\left(k^L, z'\right)\ , u\left(zk^a - k^H\right) + \beta \Gamma_{zz'} V\left(k^H, z'\right)\right\}$$

We can't say much about the choice of k', unless we have more details about the structure of the transition matrix Γ. In any case the decision rule will be a function of the form $k' = g(k,z)$.

The probability we are looking for is :

$$\Pr\{(k',z') = \left(k^L,z'\right) \mid \left(k^L,z^2\right)\} = \sum_{j=1}^{N} \Gamma_{2\ j}\{g\left(\left(k^L,z^2\right) = k^L\right)$$

Next we need to talk about stationary distribution in this model. Note that the concept of a stationary distribution here is going to be completely different compared to the Ayiagari Economy. In the latter, $x^*(B)$ is the measure of

people that belong in the subset B of the state space. Here, however, the agents are homogeneous. So, letting B be a subset of the (finite) state space, $x^*(B)$ is the probability that the economy will be in that particular set in the long run. What do we mean by "long run"? A sufficient period of time so that the initial conditions have been forgotten (do not matter any more), say 1000 periods.

Let $x^*\left(k^i,z^j\right)$ be the joint density, i.e the probability that in 1000 periods the economy will be at the state (k^i,z^j), $i = L,H$ and $j = 1,2,...,N$. Then the mean of output is

$$\bar{y} = \sum_{i=L,H} \sum_{j=1}^{N} z^j \left(k^i\right)^a x^*\left(k^i,z^j\right)$$

and the standard deviation

$$\text{SD(y)}= \sum_{i=L,H} \sum_{j=1}^{N} \left(z^j\left(k^i\right)^a - \bar{y}\right)^2 x^*\left(k^i,z^j\right).$$

Finally we have to find an expression for the stationary distribution. This is given by

$$x^*(B) = \sum_{i=L,H} \sum_{j=1}^{N} Q(s,B)\ x^*\left(k^i,z^j\right),$$

where s is the current state and B is any subset of the state space.

The connection between $x^*(B)$ and $x^*\left(k^i,z^j\right)$ is the following: recall that B is a subset of the state space, so every B is related to some pairs (i,j) indicating the state of the capital and the shock. For example if $B = \{\left(k^L,z^2\right),\left(k^L,z^3\right),\left(k^H,z^{N-1}\right)\}$, let $\mathcal{B} = \{\ (L,2),(L,3),(H,N-1)\}$. Then

$$x^*(B) = \sum_{(i,j)\in\mathcal{B}} x^*\left(k^i,z^j\right)$$

Solution 6 *The cost function is*

$$\Omega(V) = \min_{c,a,V^u} \{c + \beta(1-p(a))\Omega(V^u)\}$$

$$s.t: \; V = u(c) - a + \beta p(a) V^E + \beta(1-p(a))V^u$$

or

$$\Omega(V) = \min_{c,a,V^u} \left\{c + \beta(1-p(a))\Omega(V^u) + \theta\left[V - u(c) + a - \beta p(a) V^E - \beta(1-p(a))V^u\right]\right\}$$

Derive the FOCs:

$$\{c\}: \; 1 - \theta u_c(c) = 0 \qquad (1)$$

$$\{a\}: \; -\beta p'(a)\Omega(V^u) + \theta - \theta\beta p'(a)\left(V^E - V^u\right) = 0 \qquad (2)$$

$$\{V^u\}: \quad \beta(1-p(a))\Omega'(V^u) = \beta\theta(1-p(a)) \quad or \quad \Omega'(V^u) = \theta \qquad (3)$$

Let the solution to this problem be of the form $a = a(V)$, $c = c(V)$, $V^u = g(V)$. The envelope condition is:

$$\Omega'(V) = c'(V) - \beta p'(a)\Omega(v^u)a'(V) + \beta(1-p(a))\Omega'(V^u)g'(V) + \theta -$$

$$-\theta u_c(c)c'(V) + \theta a'(V) - \theta\beta V^E p'(a)a'(V) - \theta\beta(1-p(a))g'(V) + \theta\beta V^u p'(a)a'(V)$$

or taking the terms $c'(V), a'(V), g'(V)$ as common factors

$$\Omega'(V) = \theta + c'(V)[1-\theta u_c(c)] + a'(V)\left[\theta\beta V^u p'(a) - \theta\beta V^E p'(a) + \theta - \beta p'(a)\Omega(v^u)\right] +$$

$$+ g'(V)[\Omega'(V^u) - \theta]\beta(1-p(a))$$

or using $(1), (2)$ *and* (3)

$\Omega'(V) = \theta$, as we claimed in class.

Solution 7

Showing the convexity or strict convexity of the cost function of the insurer is complicated primarily due to a possible nonconvexity of the promise keeping constraint. Basically, the cost function is strictly convex in promised utility since efficiency implies increasing V is possible through increased consumption. Due the strict concavity of the utility, i.e decreasing MU, additional utility to be provided to the agent becomes more costly as consumption rises. Interested ones can refer to preliminary online version of the 2^{nd} edition of Ljungqvist and Sargent's Recursive Macroeconomic Theory, chapter 21 for a discussion.