Homework 4: Solving the Lifecycle Model

Econ 8503

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Baseline lifecycle model

- Recursive formulation of agent's problem:

$$V_i(a) = \max_{c,a'} \; u(c) + \beta V_{i+1}(a')$$

$$\text{s.t.} \quad c + a' = Ra + w\epsilon_i$$

$$V_{I+1}(a) = 0$$

- Constant gross interest rate R and wage w.
- Efficiency ϵ_i changes deterministically with age i.
- Agent dies at age $I + 1$.

Parameterization

- $\beta = 1$, $R = 1.03$, $w = 1$.
- Agent born at age 16, dies at age 91.
- CRRA preferences with risk aversion of 2.
- To get ϵ_i, fit quadratic to data from Minneapolis Fed QR Vol 26 No 3.
- Scale resulting coefficients by $1/5200$ to normalize time endowment to 1.
- Calculate ϵ_i as

$$\epsilon_i = a_0 + a_1 i + a_2 i^2.$$

Solution methods

- Three different solution methods: forward induction, backward induction, brute force.
- Each method takes a different approach to solving system of Euler equations (with budget constraints substituted in):

$$u_c(Ra_i + w\epsilon_i - a_{i+1}) = \beta R u_c(Ra_{i+1} + w\epsilon_{i+1} - a_{i+2}) \; \forall i = 1, \ldots, I-1$$

- Since $a_1 = a_{I+1} = 0$, this is a system of $I-1$ equations and $I-1$ unknowns $(a_2, \ldots, a_I)$.

Method 1: Forward induction

- Solve Euler equation for a_{i+2} in terms of a_{i+1} and a_i:

$$a_{i+2} = Ra_{i+1} + w\epsilon_{i+1} - u_c^{-1}\left[\frac{1}{\beta R}u_c\left(Ra_i + w\epsilon_i - a_{i+1}\right)\right]$$

- Since $a_1 = 0$, plugging in guess for a_2 gives $a_3(a_2)$.
- Similarly, using a_2 and $a_3(a_2)$ gives $a_4(a_2)$.
- Iterating forward through the system of equations in this manner gives entire asset sequence through $a_{I+1}(a_2)$.
- If we can find a_2^* such that $a_{I+1}(a_2) = 0$, we've solved the model.

How do we find a_2^*?

- Iteration process on previous slide defines function $a_{I+1} : \mathbb{R} \to \mathbb{R}$.
- Finding a_2^* is equivalent to root-finding problem in one dimension.
- Writing down analytical derivative of $a_{I+1}(a_2)$ is hard, so can either use derivative-based method with numerical derivatives or bracketing method.
- Bisection code from homework 3 would work fine!

Method 2: Backward induction

- Same basic principle as forward induction, but we guess a_I instead and iterate backward, looking for a_I^* such that $a_1(a_I^*) = 0$.
- Solve Euler equation for a_i:

$$a_i = \frac{1}{R}\left\{u_c^{-1}\left[\beta R u_c\left(Ra_{i+1} + w\epsilon_{i+1} - a_{i+2}\right) + a_{i+1} - w\epsilon_i\right]\right\}$$

- Since $a_{I+1} = 0$, plugging in guess for a_I gives $a_{I-1}(a_I)$. Iterate backwards to get $a_1(a_I)$.
- Again, we have one-dimensional function $a_1 : \mathbb{R} \to \mathbb{R}$ of which we need to find the root.

Method 3: Brute force

- Basic idea: Consider system of Euler equations as single multidimensional function. Find its root!
- Define $F : \mathbb{R}^{I-1} \rightarrow \mathbb{R}^{I-1}$ by

$$F_i(a_2, \ldots, a_I) = \beta R u_c \left[R a_{i+1} + w\epsilon_{i+1} - a_{i+2} \right] - u_c \left[R a_i + w\epsilon_i - a_{i+1} \right]$$

- Since $a_1 = a_{I+1} = 0$, F_1 and F_{I-1} only have two unknowns each:

$$F_1(a_2, \ldots, a_I) = \beta R u_c \left[R a_2 + w\epsilon_2 - a_3 \right] - u_c \left[R(0) + w\epsilon_1 - a_2 \right]$$
$$F_{I-1}(a_2, \ldots, a_I) = \beta R u_c \left[R a_I + w\epsilon_I - 0 \right] - u_c \left[R a_{I-1} + w\epsilon_{I-1} - a_I \right]$$

Brute force continued

- To find root $x^* = (a_2^*, \ldots, a_I^*)$ such that $F_i(x^*) = 0, \ \forall i$, use Newton-Raphson.
- Start with initial guess $x^0 = (a_2^0, \ldots, a_I^0)$ and obtain x^1 by

$$x^1 = x^0 - J^{-1}(x^0)F(x^0)$$

  where $J(x)$ is Jacobian matrix of F evaluated at x.
- Obtain $x^2, x^3, \ldots$ in same manner. Stop when x^n converges.
- Caution: Newton-Raphson is unstable and sensitive to initial guess. Try several guesses to be see if they give the same result.

Results

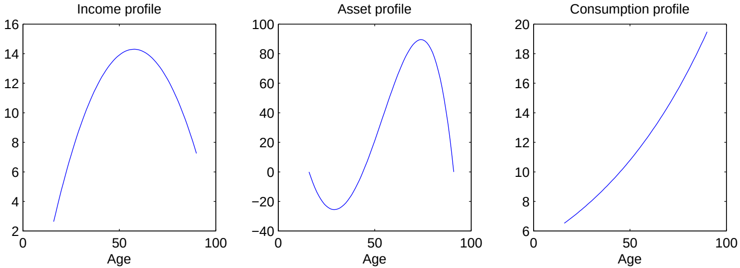

Figure 1: Results for baseline lifecycle model

Adding endogenous labor supply

- To make the consumption profile U-shaped we can add leisure to the utility function in a nonseparable way:

$$u(c, 1-n) = \frac{\left(c^{\eta}(1-n)^{1-\eta}\right)^{1-\sigma}}{1-\sigma}.$$

- Equations that characterize the solution are now

$$u_c(c_i, 1-n_i) = \beta R u_c(c_{i+1}, 1-n_{i+1})$$
$$c + a_{i+1} = Ra_i + wn_i\epsilon_i$$
$$-\frac{u_\ell(c_i, 1-n_i)}{u_c(c_i, 1-n_i)} \geq w\epsilon_i, \quad \text{``='' if } n_i > 0$$

- Parameter additions/changes: $\beta = 0.97$, $\eta = 0.6$.

How do we solve model with endogenous labor?

- Goal: reduce to system of second-order difference equations.
- FOC for leisure implies $n_i = \max\left[1 - \frac{1-\eta}{\eta}\frac{c_i}{w\epsilon_i}, 0\right]$.
- This means four different possibilities for the Euler equation:

$$\begin{cases} c_i^{-\sigma} = \beta R \left[\left(\frac{\epsilon_i}{\epsilon_{i+1}}\right)^{(1-\eta)(1-\sigma)}\right] c_{i+1}^{-\sigma} & \text{if } n_i > 0,\ n_{i+1} > 0 \\ \left[\left(\frac{1-\eta}{\eta w \epsilon_i}\right)^{(1-\eta)(1-\sigma)}\right] c_i^{-\sigma} = \beta R c_{i+1}^{\eta(1-\sigma)-1} & \text{if } n_i > 0,\ n_{i+1} = 0 \\ c_i^{\eta(1-\sigma)-1} = \beta R \left[\left(\frac{1-\eta}{\eta w \epsilon_{i+1}}\right)^{(1-\eta)(1-\sigma)}\right] c_{i+1}^{-\sigma} & \text{if } n_i = 0,\ n_{i+1} > 0 \\ c_i^{\eta(1-\sigma)-1} = \beta R c_{i+1}^{\eta(1-\sigma)-1} & \text{if } n_i = 0,\ n_{i+1} = 0 \end{cases}$$

Solving model with endogenous labor continued

- Can apply same three solution methods from baseline model, but they're more complicated now.
- For each $i = 1, \ldots, I-1$, have to check if $n_i = 0$ and $n_{i+1} = 0$ to see which Euler equation applies.
- I will show forward induction algorithm that uses hint from Jan and Tayyar.
- Basic idea: Guess c_1 rather than a_2 and iterate on the consumption sequence. This makes handling the different Euler equations easier.

Algorithm for forward induction with endogenous labor

1. Guess c_1.
2. Calculate $n_1(c_1) = \max\left[1 - \frac{1-\eta}{\eta}\frac{c_1}{w\epsilon_0}, 0\right]$ and use budget constraint with $a_1 = 0$ to find $a_2(c_1)$.
3. Guess that $n_2(c_1) > 0$ and calculate $c_2(c_1)$ using first or the third Euler equation (depends on whether $n_1(c_1) > 0$).
4. Calculate $n_2(c_1) = \max\left[1 - \frac{1-\eta}{\eta}\frac{c_2(c_1)}{w\epsilon_1}, 0\right]$. If $n_2(c_1) = 0$, recalculate $c_2(c_1)$ using the second or fourth Euler equation.
5. Repeat steps 2-4 for each Euler equation in system until $c_I(c_1)$ and $a_I(c_1)$ are obtained.
6. Calculate $a_{I+1}(c_1)$ using calculated value of $c_I(c_1)$ in budget constraint for period I.
7. Use bracketing method to find c_1^* such that $a_{I+1}(c_1^*) = 0$.

Results after adding endogenous labor

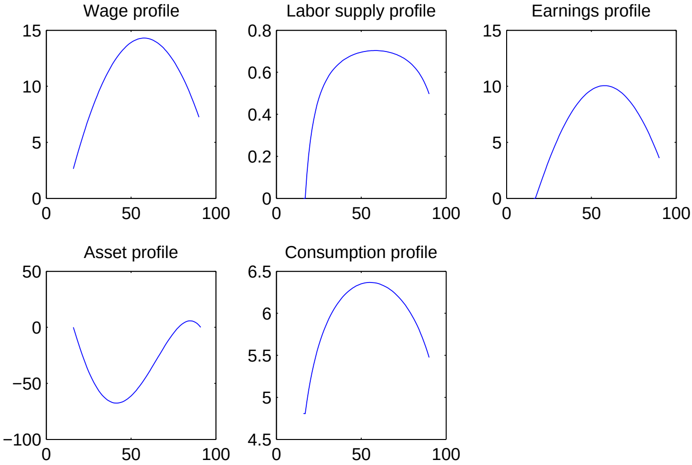

Figure 2: Results for lifecycle model with leisure (age on x-axis)