Macro Het Agents 081

Preliminary

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Measure Theory

Measure theory is a tool that helps us aggregate.

Definition

For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).

Remark

Note that in this section we will assume the following convention

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) are for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition

A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if

1. $S, \emptyset \in \mathcal{S}$;
2. if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,
3. for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections.

Example

1. The power set of S and $\{\emptyset, S\}$ are σ-algebras in S.
2. $\left\{\emptyset, S, S_{1/2}, S_{2/2}\right\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.
3. If $S = [0, 1]$, then $\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$ is *not* a σ-algebra in S. But $\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$ is.

It allows us to define sets where things happen and we can *weigh* those sets (avoiding math troubles)

Definition

Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies

1. $x(\emptyset) = 0$;
2. if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,
3. if $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).

A set S, a σ-algebra in it ($\mathcal{S}$), and a measure on $\mathcal{S}$ x, define a measurable space, $(S, \mathcal{S}, x)$.

Definition

A Borel σ-algebra is a σ-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.

A Borel σ-algebra corresponds to complete information.

Definition

A probability measure is measure where $x(S) = 1$. $(S, \mathcal{S}, x)$ is a probab space. The probab of an event is then given by $x(A)$, where $A \in \mathcal{S}$.

Definition

Given a m'able space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is m'able (with respect to the m'able space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Interpret σ-algebras as describing available information.

Similarly, a function is m'able wrt a σ-algebra $\mathcal{S}$, if it can be evaluated

Example

Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function f that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.

Extend the notion of Markov stuff to any measurable space

Definition

Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if

1. $Q(s, \cdot)$ is a probability measure for all $s \in S$; and,
2. $Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set B tomorrow, given that the state is s today.

1. A *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on $S = \{1, 2, 3\}$, with the the power set being the σ-algebra of S).

$$Q(3, \{1, 2\}) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 \ .$$

2. Consider a measure x on $\mathcal{S}$. x_i is the fraction of type i. Then

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words: $\mathrm{x}' = \Gamma^T \mathrm{x}$, where $\mathrm{x}^T = (x_1, x_2, x_3)$.

From a measure x today to one tomorrow x'

$$\begin{aligned} x'(B) &= T(x, Q)(B) \\ &= \int_S Q(s, B)\, x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

we integrated over all $s \in S$ to get the prob of B tomorrow.

A stationary distribution is a fixed point of T, that is x^* such that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

Theorem

If Q has nice properties (American Dream and Nightmare) then $\exists$ a unique stationary distribution x^* and

$$x^* = \lim_{n \to \infty} T^n(x_0, Q), \qquad \textit{for any } x_0.$$

Exercise

Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

Industry Equilibrium

- Study the dynamics of the distribution of firms in partial equilibrium
- A single firm produces a good using labor:
- Output is $sf(n)$ (f increasing, strictly concave, $f(0) = 0$, s is productivity.
- Markets are competitive, (p and $w = 1$) as given.
- A firm solves

$$\pi(s, p) = \max_{n \geq 0} \{psf(n) - wn\}. \quad (1)$$

- With FOC

$$psf_n(n^*) = 1. \quad (2)$$

  Solution is $n^*(s, p)$.

- n^* is an increasing function of both arguments. Prove it.

- A mass of firms in the industry, indexed by $s \in S \subset \mathbb{R}_+$, $S := [\underline{s}, \bar{s}]$.
- $\mathcal{S}$ is a σ-algebra on S (a Borel σ-algebra, for instance).
- x is a measure on $(S, \mathcal{S})$, which describes the cross-sectional distribution of productivity among firms.
- Use x to define statistics of the industry: Since individual supply is $sf\left(n^*\left(s, p\right)\right)$, then the aggregate supply

$$Y^S\left(p\right) = \int_S sf\left(n^*\left(s, p\right)\right) x\left(ds\right). \tag{3}$$

Y^S is a function of the price p only.

- Let Demand $Y^D\left(p\right)$. Then p^* clears the market:

$$Y^D\left(p^*\right) = Y^S\left(p^*\right). \tag{4}$$

Where does x come from?

- Price p and output Y are constant over time.
- Firms face the problem above every period and discount profits at exogenous r.
- Each firm faces a probability $1-\delta$ of disappearing in each period.
- The choice is static. The value of an s firm is

$$V(s;p) = \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r}\right)^t \pi(s,p) = \left(\frac{1+r}{1+r-\delta}\right)\pi(s,p)$$

- Every period a mass of firms die. To achieve a stationary equilibrium we need firms entry: assume that there is a constant flow of firms entering the economy in each as well, so that entry equals exit.
- x is the measure of firms. Firms that die are $(1-\delta)x(S)$.
- Entrants draw s from probability measure γ over $(S,\mathcal{S})$.

- What keeps other firms out of the market in the first place?
- (if $\pi(s;p) = psf(n^*(s;p)) - wn^*(s;p) > 0$, then any firm with $s \in S$ would enter.
- Assume a fixed entry cost, c^E before learning s. Value of an entrant

$$V^E(p) = \int_S V(s;p)\,\gamma(ds) - c^E. \tag{5}$$

If $V^E > 0$ there will be entry.

- Equilibrium requires $V^E = 0$

- x_t : cross-sectional distribution of firms. For any $B \subset S$, fraction $1 - \delta$ of firms with $s \in B$ die and mass m of newcomers enter. Next period's measure of firms on set B is

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B). \tag{6}$$

- Mass m of firms would enter $t + 1$, with fraction $\gamma(B)$ having $s \in B$, $\forall B \in \mathcal{S}$.

- Cross-sectional distribution of firms completely determined by γ.

- Consider an updating operator T

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{7}$$

  a stationary dbon is a fixed point, i.e. x^* such that $Tx^* = x^*$, so

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{8}$$

- Demand and supply condition in equation (4) is

$$Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right), \tag{9}$$

whose solution $p^*\left(m\right)$ is a continuous function

- We have two equations, (5) and (9), and two unknowns, p and m.

Definition

A stationary distribution for this environment consists of functions V, π^*, n^*, p^*, x^*, and m^*, that satisfy:

1. Given prices, V, π^*, and n^* solve the incumbent firm's problem;
2. $Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right)$;
3. $\int_s V\left(s;p\right)\gamma\left(ds\right) - c^E = 0$; and,
4. $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$.

- Assume s follows a Markov process with transition Γ. This would change the mapping T in Equation (7) to

$$Tx(B) = \delta \int_S \Gamma(s, B)\, x(ds) + m\gamma(B), \quad \forall B \in \mathcal{S}. \tag{10}$$

But no firm exits (c^E is a sunk cost). Still not much Econ.

- Suppose now an operating cost c^v each period.
  - when s is low, firm's profits maybe negative and firm exits
  - But it is not enough. Assume Γ satisfies stochastic dominance: $s^1 > s^2$ implies $\sum_{s'=1}^{\widehat{s}} \Gamma_{s^1,s'} < \sum_{s'=1}^{\widehat{s}} \Gamma_{s^2,s'}$.
  - Then $\exists$ a threshold, $s^* \in S$, below which firms exit and above stay.

$$V(s;p) = \max\left\{0, \pi(s;p) + \frac{1}{(1+r)} \int_S V(s';p)\, \Gamma(s, ds') - c^v\right\}. \tag{11}$$

- Updating operator becomes

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}. \tag{12}$$

A stationary distribution of the firms in this economy, x^*, is the fixed point of this equation.

- With x^* we get all class of statistics:
    - Threshold for being in top 10% by size? Solve for $\widehat{s}$

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1,$$

    - Fraction of workers in largest top 10% of firms

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^*(ds)}.$$

Do

Exercise

Compute the average growth rate of the smallest one third of the firms.

Exercise

What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?

Exercise

What is the fraction of firms younger than five uears?

Definition

π^*, n^*, d^*, s^*, V, a price p^*, a measure x^*, and mass m^* such that

1. Given p^*, the functions V, π^*, n^*, d^* solve the firm's
2. The reservation productivity s^* satisfies $d^*(s;p^*) = \begin{cases} 1 & \text{if } s \geq s^* \\ 0 & \text{otherwise} \end{cases}$.
3. Free-entry condition: $V^E(p^*) = 0$.
4. For any $B \in \mathcal{S}$

$$x^*(B) = m^*\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x^*(ds)$$

5. Market clearing:

$$Y^d(p^\star) = \int_{s^\star}^{\bar{s}} s\ f(n^\star(s;p^\star))x^\star(ds)$$

- Average output of the firm is given by

$$\frac{Y}{N} = \frac{\int_{s^\star}^{\bar{s}} s \; f[n^*(s)] \; x^*(ds)}{\int_{s^\star}^{\bar{s}} x^*(ds)}$$

- Share of output produced by the top 1% of firms. Need to find $\tilde{s}$

$$\frac{\int_{\tilde{s}}^{\bar{s}} x^*(ds)}{\int_S x^*(ds)} = .01$$

$$\frac{\int_{\tilde{s}}^{\bar{s}} s \;\; f[n^*(s)] \;\; x^*(ds)}{\int_{s^\star}^{\bar{s}} s \;\; f[n^*(s)] \;\; x^*(ds)}$$

- Fraction of firms in the top 1% two periods in a row (s continuous)

$$\int_{s \geq \tilde{s}} \int_{s' \geq \tilde{s}} \Gamma_{ss'} x^*(ds)$$

- Gini coefficient.

Consider adjustment costs to labor $c\left(n^{-}, n\right)$ due to hiring n units of labor in t as

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha\left(n_t - n_{t-1}\right)^2$ units of the numeraire good. The cost here depends on the size of the adjustment.

- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2$ units of the numeraire good only if $n_t > n_{t-1}$. We can write this as

$$1_{\left\{n_t > n_{t-1}\right\}} \alpha\left[n_t - (1-\delta)\, n_{t-1}\right]^2,$$

where 1 is the indicator function and δ measures the exogenous attrition of workers in each period.

- *Firing Costs*: the firm has to pay if it wants to reduce the number of workers.

$$V\left(s,n^{-};p\right)=\max\left\{0,\max_{n\geq 0}\ sf\left(n\right)-wn-c^{v}-c\left(n^{-},n\right)+\frac{1}{(1+r)}\int_{s'\in S}V\left(s',n,p\right)\ \Gamma(s,ds')\right\},$$

$c(\cdot,\cdot)$ is cost function (note limited liability: exit value is 0)

Note $n=g(s,n^{-};p)<\bar{N}$. Let $\mathcal{N}$ be a σ-algebra on $[0,\bar{N}]$.

$$x'\left(B^{S},B^{N}\right)=m\gamma\left(B^{S}\cap[s^{*},\bar{s}]\right)1_{\{0\in B^{N}\}}+\int_{s^{*}}^{\bar{s}}\int_{0}^{\bar{N}}1_{\{g(s,n_{-};p)\in B^{N}\}}\ \Gamma\left(s,B^{S}\cap[s^{*},\bar{s}]\right)x\left(ds,dn_{-}\right),$$

$$\forall\ B^{S}\in\mathcal{S},\forall\ B^{N}\in\mathcal{N}.$$

Exercise
Write the first order conditions.

Exercise
Define the recursive competitive equilibrium for this economy.

Exercise
Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to function c. The firm also pays a cost κ to post vacancies and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?

Exercise
Add Adjustment Costs to Capital

Exercise
Add R& D

- So far *stationary industry equilibria* (invariant distribution of firms).
- If p were constant, the firm distribution would converge to the stationary equilibrium distribution x^*.
- What is an alternative?
- Prices are changing over time and so is the distribution of firms.
- There are two ways of modeling non-stationary equilibria
    - In Sequence Space (or stochastic process state)
    - Recursively
- What is best depends on the purpose. They should give the same answer. It is an issue of computation.
- We will look at both ways (for now deterministic).
- Given the convergence that we talked about we need a rationale for the non stationarity.
- Consider demand shifters z_t so that $D(P, z_t)$ where $z_{t+1} = \phi(z_t)$ so we can choose to represent it as a sequence or recursively.

- Note the need for an initial condition. Then objects are relatively simple.
- Given a path $\{z_t\}_{t=0}^{\infty}$ and an initial x_0, an equilibrium defined in term of sequences is: Sequences $\{p_t, m_t, s_t^*\}$ of numbers, a sequence of measures x_t, and sequences $\{V_t(s), n_t(s)\}_{t=0}^{\infty}$ of functions:

1. **Optimality**: Given $\{p_t\}$, $\{V_t, s_t^*, n_t\}$ sole

$$V_t(s) = \max \left\{ 0, \max \ p_t \ s \ f(n) - wn - c^v + \frac{\int_S V_{t+1}(s') \Gamma(s, ds')}{1+r} \right\}$$

2. **Free-entry**: $\int V_t(s) \ \gamma(ds) \le c^e$, with strict equality if $m_t > 0$.
3. **Law of motion**: $x_{t+1}(B) = m_{t+1}\gamma\left(\cap[s_{t+1}^*, \bar{s}]\right) + \int_{s_t^*}^{\bar{s}} \Gamma\left(s, B \cap [s_{t+1}^*, \bar{s}]\right) \ x_t(ds)$, $\forall B \in \mathcal{S}$.
4. **Market clearing**: $D[p_t, z_t) = \int_{s_t^*}^{\bar{s}} p_t \ s \ f[n_t(s)] \ x_t(ds)$.

- Only from today to tomorrow: need objects that given the state today, $\{z, x\}$, give us the state tomorrow $\{\phi, G\}$.
- Given ϕ, an equil defined recursively is functions $G(z, x)$, $m(z, x)$, $p(z, x)$, values and decisions $\{V(s, z, x), n(s, z, x), s^*(s, z, x)\}$ s.t.

1. **Optimality**: $\{V(s, z, x), s^*(s, z, x), n(s, z, x)\}$ solve

$$V(s, z, x) = \max_n \left\{0, \max p(s, z, x)s\ f(n) - wn - c^v + \frac{1}{1+r} \int_S V[s', \phi(z), G(z, x)]\ \Gamma(s, ds')\right\}$$

2. **Free-entry**: $\int V(s, z, x)\ \gamma(ds) \leq c^e$, $(=$ if $m(z, x) > 0)$.

3. **Law of motion**: $\forall B \in \mathcal{S}$, we have
$G(z, x)(B) = m(z, x)\ \gamma(B \cap [s^*(s, z, x), \bar{s}]) + \int_{s^*(s,z,x)}^{\bar{s}} \Gamma(s, B \cap [s^*(s, z, x), \bar{s}])x(ds)$,

4. **Market clearing**: $D(p(z, x), z) = \int_{s^*(s,z,x)}^{\bar{s}} p(z, x)\ s\ f[n(s, z, x)]\ x(ds)$.

- It is the same but in Stochastic Processes Language

- They extend the same for sequences and for the Recursive

- Obviously You have to add the Expectations to the terms of one period later.

- There is a new (Boppart, Mitman & Krusell (2017)) way of thinking of Stochastic Equilibria that is NOT recursive.
- It is based on a linear approximation to a completely unanticipated (MIT) shock.
- It requires to compute a transition as a Perfect Foresight Equilibrium
- Then do linear approximations in sequence space.

- Consider the social planner's problem (with full depreciation)

$$
\begin{aligned}
V(k_t) \quad = \quad & \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \\
& \text{s.t. } c_t + k_{t+1} \leq f(k_t), \qquad \forall\ t \geq 0 \\
& c_t, k_{t+1} \geq 0, \ \forall \qquad t \geq 0 \\
& k_0 > 0 \text{ given.}
\end{aligned}
$$

- The solution $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ satisfies

$$u_c(c_t) = \beta\ u_c(c_{t+1})\ f_k(k_{t+1}), \forall\ t \geq 0$$

$$c_t + k_{t+1} = f(k_t), \quad \forall\ t \geq 0$$

$$\lim_{t \to \infty}\ \beta^t\ \ u_c(c_t)\ \ k_{t+1} = 0$$

- Derive the above equilibrium conditions.

- Look at the a steady state k^*
- Rewrite solution as

$$\psi(k_t, k_{t+1}, k_{t+2}) = u_c[f(k_t - k_{t+1})] - \beta\ u_c[f(k_{t+1} - k_{t+2})]\ f_k(k_{t+1}) = 0,$$

a second order difference equation with exactly two boundary conditions, k_0 and $k_\infty = k^*$.

- It is solvable:
  1. guess k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some T, and solve $k_T^{\psi}(k_1) = k^*$.
  2. Or guess k_{T-1} solve backward using ψ to find $k_0^{\psi}(k_{t-1}) = k_0$
  3. Solve for the whole sequence as a system of equations (almost diagonal)
  4. Use dynare.
- Either way you get a numerical solution starting from any k_0

- We can compute any transition. Also one with time varying ψ.
- Consider this model with $c_t + k_{t+1} = e^{z_t}\ f(k_t),\ z_{t+1} = \rho z_t,\ z_0 = 1$.

$$\psi_t(k_t, k_{t+1}, k_{t+2}) = u_c[\rho^t f(k_t - k_{t+1})] - \beta u_c[\rho^{t+1} f(k_{t+1} - k_{t+2})] f_k(k_{t+1}),$$

- In this case we can look at an MIT shock or impulse response. Here $k_0 = k_\infty = k^*$, but $k_1 \neq k^*$
- We can again obtain the transition k_t.
- Let now $\widehat{k}_t = \log k_t - \log k^*$, (log st st deviation).
- This is in fact an impulse response function.

- We want now to simulate a response of the economy to shocks. Consider an AR(1) process for z_t : with $z_{t+1} = \rho^t z_t + \epsilon_{t+1}$.) where $\epsilon_t \sim \mathcal{N}(\int, \supset^{\in})$.
- **Want:** Solve for the solution by linearly approximating using $\{\widehat{k}_t\}_{t=0}^{\infty}$ the equilibrium given any sequence of innovations $\{\epsilon_t\}$.).
- Obtain $\widetilde{k}_t(k_0, \epsilon^{t-1})$ again in deviations from steady state. Note that the following linear approximation is what we want.

$$
\begin{aligned}
\widetilde{k}_1(k_0, \epsilon_0) &= \epsilon_0 \ \widehat{k}_1 \\
\widetilde{k}_2(k_0, \epsilon_0, \epsilon_1) &= \epsilon_0 \ \widehat{k}_2 + \epsilon_1 \ \widehat{k}_1, \\
&\vdots \\
\widetilde{k}_{t+1}(k_0, \epsilon^t) &= \sum_{\tau=0}^{t} \epsilon_t \ \widehat{k}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = 1, \ \epsilon_t = 0, \forall t \neq 0,
\end{aligned}
$$

- This can be done for all Economies.
- Including industry equilibria.
- For all Statistics of all Economies.
- The computational costs is linear not exponential in the number of shocks.
- We do not know how to use it for asymmetric shocks (e.g. downward rigid wages)

Exercise

1. *What happens if demand suddenly doubles starting from a stationary equilibrium?*
2. *Define Formally the stochastic counterparts (sequentially and recursivrly) to the ones that we wrote above?*
3. *Sketch an algorithm to find the equilibrium prices.*
4. *Describe a way to compute the evolution of the Gini Index or the Herfindahl Index of the industry over the first fifteen periods.*
5. *Imagine now that the industry is subject to demand shocks that follow an AR(1). Describe an algorithm to approximate it.*

Incomplete Market Models

- Consider the problem of a farmer with storage possibilities

$$V(s,a) = \max_{c,a' \geq 0} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} \; V(s', a') \qquad s.t.$$

$$c + qa' = a + s$$

a assets, c consumption, and $s \in \{s^1, \cdots, s^{N^s}\} = S$ has transition Γ. q units today yield 1 unit tomorrow. Only nonnegative storage.

- If s constant, then

$$V(a) = \max_{c, a' \geq 0} \left\{ u\left(a + s - qa'\right) + \beta V\left(a'\right) \right\}.$$

- with FOC $q\, u_c \geq \beta u_c'$
- With equality if $a' > 0$. Then
  - if $q > \beta$ (i.e. nature is more stingy, or the farmer is less patient),
    - Either $c' < c$ and the farmer dis-saves
    - Or $c = s$ and $a' = 0$.
  - If $q < \beta$, $c' > c$ and consumption grows without bound.
  - If $q = \beta$, $c' = c$ (with noise and $u_{ccc} > 0$ grows without bound).
- So we assume $\beta/q < 1$

- Assuming $\beta/q < 1$, allows us to bound asset holdings.
- They also save in best states when a is low.
- The FOC is

$$u_c\left[c\left(s,a\right)\right] \geq \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}\ u_c\left(c\left[s', g\left(s,a\right)\right]\right),$$

with equality when $a' = g\left(s,a\right) > 0$

- Note: $a \gg g\left(s,a\right), \ \ \forall s$ for sufficiently large a. So $\exists\ \bar{a},\ s.t.\ a' \in A = [0,\bar{a}]$
- We can construct a prob distribution over states $S \times A$. Define $\mathcal{B}$ as all subsets of S times Borel-σ-algebra sets in A.
- For any such prob measure x its evolution is

$$x'\left(B\right) = \widetilde{T}(B,x;\Gamma,g) = \sum_{s}\int_{0}^{\bar{a}}\sum_{s'\in B_s}\Gamma_{ss'}\ 1_{\{g(s,a)\in B_a\}}\ x\left(s,da\right), \quad \forall B\in\mathcal{B}$$

where B_s and B_a are projections of B on S and A,

Theorem

With a well behaved Γ, there is a unique stationary probability x^*, so that:

$$x^*(B) = \widetilde{T}(B, x^*; \Gamma, g)(B), \quad \forall B \in \mathcal{B},$$
$$x^*(B) = \lim_{n \to \infty} \widetilde{T}^n(B, x_0; \Gamma, g)(B), \quad \forall B \in \mathcal{B},$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

We use compactness of $[0, \overline{A}]$.

1. Our ignorance of what is going on with the farmer or fisherman.

   - Even if we know at $t = 0$ s, a, no news lead us to x^*.

   - We can use x^* to compute the statistics of what happens to the fisherman: Average wealth is $\int_{S \times A} a \, dx^*$.

2. A description of a large number of fishermen (an archipelago). Notice how even if there is no contact between them. Stationarity arises (İmrohoroğlu (1989))

   - There is a unique distribution of wealth.

- How can $a < 0$? Because of borrowing.
- Consider now an economy with many farmers and NO storage.

$$V(s,a) = \max_{c \geq 0, a'} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a')$$
$$s.t. \quad c + q\, a' = a + s$$
$$a' \geq \underline{a},$$

where $\underline{a} < 0$ and $\beta / q < 1$. With solution $a' = g(s,a)$. Again

- One possibility for $\underline{a}$ is the natural borrowing limit: the agent can pay back his debt with certainty, no matter what:

$$a^n := -\frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}. \tag{13}$$

- Or it could be tighter which makes it likely to bind $0 > \underline{a} > a^n$.

- To determine q in general equilibrium, consider this function of q:

$$\int_{A\times S} a\, dx^*(q) \qquad \text{Aggregate asset holdings}$$

- A Stationary Equilibrium requires two things

$$\begin{aligned} \int_{A\times S} a\, dx^*(q) &= 0, \\ x^*(q) &= \widetilde{T}^n\left(B, x^*(q); \Gamma, g\right)(B). \end{aligned}$$

- It exists in $q \in (\beta, \infty]$ (intermediate value thm). Need to ensure:
  1. $\int_{A\times S} a\, dX^*(q)$ is a continuous function of q;
  2. $\lim\limits_{q\to\beta} \int_{A\times S} a\, dX^*(q) \to \infty$; (As $q \to \beta$, the interest rate $R = 1/q$ increases up to $1/\beta$, (steady state interest rate in deterministic Econ. With $u_{ccc} > 0$ we have precautionary savings
  3. $\lim\limits_{q\to\infty} \int_{A\times S} a\, dX^*(q) < 0$. As $q \to \infty$, arbitrary large consumption is achievable by borrowing.

- Workhorse models of modern macroeconomics.
- An Environment like the ones before
- On top of a growth model with $f(K, L)$ that yield factor prices.

$$K = \int_{A\times S} a\,dx,$$
$$N = \int_{A\times S} s\,dx.$$

- s fluctuations in the employment status (either efficiency units of labor or actual employment).
- Now we need $\beta(1+r) < 1$. We write

$$V(s,a) = \max_{c,a'\geq 0} \quad u(c) + \beta \int_{s'} V(s',a')\ \Gamma(s, d\,s') \qquad s.t.$$
$$c + a' = (1+r)\,a + ws$$

where r is the return on savings and w is the wage rate.

- Factor prices depend on the capital-labor ratio: $x^*\left(\frac{K}{L}\right)$. Equilibrium requires

$$\frac{K^*}{L^*} = \frac{\int_{A\times S} a \ dX^*\left(\frac{K^*}{L^*}\right)}{\int_{A\times S} s \, dX^*\left(\frac{K^*}{L^*}\right)}.$$

Exercise

Show that aggregate capital is higher in the stationary equilibrium of the Aiyagari economy than it is the standard representative agent economy.

Exercise

Not necessarily so if leisure has value (Pijoan-Mas (2006))

Exercise

Rewrite the economy when households like leisure

- Let the Economy's parameters be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$.
- $V(s, a; \theta)$ and $x^*(\theta)$ are functions of those parameters.
- Suppose an unexpected policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$.
- Consider $V\left(s, a; \hat{\theta}\right)$ and $x^*\left(\hat{\theta}\right)$.
- Define $\eta(s, a)$ by

$$V\left(s, a + \eta(s, a); \hat{\theta}\right) = V(s, a; \theta),$$

- Transfer necessary to make the (a, s) agent indifferent between living in the old environment and in the new.
- Total transfer needed to compensate all agents to live in $\hat{\theta}$ is

$$\int_{A \times S} \eta(s, a)\, dX^*(\theta).$$

- This is NOT a Welfare Comparison.
- This compares being parachuted in the stationary distribution of θ versus $\hat{\theta}$.
- Welfare computing the transition from the SAME initial conditions.
- Otherwise the best tax policy in the Rep agent (which is Pareto Optimal) would be to subsidize capital to maximize steady state consumption.

- What if aggregate shocks as in e.g. $z\ F\left(K, \bar{N}\right)$.
- Without leisure aggregate capital is a sufficient statistic for factor prices.
- Will aggregate capital be $K' = G(z, K)$ or $K' = G(z, x)$?
- The latter. Decision rules are not usually linear. But then $x' = G(z, x)$

$$
\begin{aligned}
V(z, X, s, a) = \max_{c, a' \geq 0} \quad & u(c) + \beta \sum_{z', s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V\left(z', X', s', a'\right) \\
s.t. \quad & c + a' = a\ z\ f_k\left(K, \bar{N}\right) + s\ z\ f_n\left(K, \bar{N}\right) \\
& K = \int a\ dX(s, a) \\
& X' = G(z, X)
\end{aligned}
$$

(replaced factor prices with marginal productivities)

- Computationally, this problem is a beast! So, what then?

- They people believe tomorrow's capital depends only on K and not on x. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$\widetilde{V}(z,K,s,a) = \max_{c,a'} \quad u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}\left(z',K',s',a'\right)$$
$$s.t. \quad c + a' = a\ z\ f_k\left(K,\bar{N}\right) + szf_n\left(K,\bar{N}\right)$$
$$K' = \widetilde{G}(z,K)$$

- We could approximate the equilibrium in the computer by posing a linear approximation to $\widetilde{G}$. A pain but doable. Krusell Smith (1997).

- They found it works well in boring settings (things are pretty linear)

- We can use the same linear approx in sequences as before for any shocks:

    1. Find the steady state

    2. Obtain the the impulse response (the perfect foresight equilibium) given an MIT shock that is treated as an innovation.

    3. Use these responses to approximate the behavior of any aggregate.

- Valuable for SMALL shocks like all linear approximations.

- Consider an Aiyagari economy with an AR(1) TFP shock z.
  - Choose an initial size innovation $\overline{\epsilon}_0$ (does not have to be 1) and compute the perfect foresight Equilibria of this MIT shock.
    - This involves a fixed point in the space of sequence of capital labor ratios.
    - But can be done with some effort:
    - To evaluate it, given prices solve the household's problem backwards from the final steady state.
    - Then update the distribution forward from the initial steady state obtaining new prices.
    - We look for a fixed point of this (not necessarily iterating mechanically but as solution of a system of equations)

- We have now the sequence of x_t and any prices that we care for.
- Compute the sequence of all statistics $\{d_t\}_t^T$ of that economy that you care for.
- Get a random draw $\{\epsilon_t\}_{t=0}^T$.
- Linearly approximate those statistic like we did before the same way that we approximated

$$\begin{aligned}
\widetilde{d}_1(x_0, \epsilon_0) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_1 \\
\widetilde{d}_2(x_0, \epsilon_0, \epsilon_1) &= \frac{\epsilon_0}{\bar{\epsilon}_0}\ \widehat{d}_2 + \frac{\epsilon_1}{\bar{\epsilon}_0}\ \widehat{d}_1, \\
&\vdots \\
\widetilde{d}_{t+1}(x_0, \epsilon^t) &= \sum_{\tau=0}^{t} \frac{\epsilon_t}{\bar{\epsilon}_0}\ \widehat{d}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = \widetilde{\epsilon}_0,\ \epsilon_t = 0, \forall t \neq 0.
\end{aligned}$$

- Agents can either not work or work: $\varepsilon = \{0, 1\}$,
- Agents can exert painful effort h to search for a job increasing the probability $\phi(h)$ (with $\phi' > 0$) of finding it.
- An employed worker, does not search for a job so $h = 0$, but its job can be destroyed with some exogenous probability δ.
- s is Markovian (Γ) labor labor productivity. Then the unemployed

$$V(s,0,a) = \max_{c,h,a' \geq 0} u(c,h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h) V(s',1,a') + (1-\phi(h)) V(s',0,a')\right]$$

$$s.t. \quad c + a' = h + (1+r)a$$

the employed

$$V(s,1,a) = \max_{c,a' \geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s',0,a') + (1-\delta) V(s',1,a')\right]$$

$$s.t. \quad c + a' = sw + (1+r)a$$

- Suppose every period agents choose an occupation: entrepreneur or a worker.

- Entrepreneurs run their own business: manage a project that combines entrepreneurial ability (η), capital (k), and labor(n); while workers work for somebody else.

- If worker

$$V^w(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'}\left[dV^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t. \quad c + a' = ws + (1+r)a$$

- Similarly, the entrepreneur's problem can be formulated as follows

$$V^e(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[d\ V^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

$$s.t. \quad c + a' = \pi(s,\eta,a)$$

- Income is from profits $\pi(a,s,\eta)$ not wages. Assume entrepreneurs have a DRS technology f. Profits are

$$\pi(s,\eta,a) = \max_{k,n} \eta f(k,n) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\}$$

$$s.t. \quad k - a \leq \phi a$$

- The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth.

- Entrepreneurs never make an operating loss within a period, (can always choose $k = n = 0$ and earn the risk free rate on savings).
- Agents with high entrepreneurial ability η have access to an investment technology f that provides higher returns than workers so become richer.
- Even the prospects (high η) low wealth suffice to induce high savings? (Γ)
- Who becomes an entrepreneur in this economy? Without financial constraints, wealth will play no role. $\exists \eta^*$ above which it becomes an entrepreneur.
- With financial constraints wealth matters. Wealthy agents with high h will while the poor with low η will not.
- For the rest, it depends. If η is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

- The price of lending incorporates the possibility of default.
- Assume upon default punished to autarky forever after (no borrowing or lending)
- If no default the budget constraint is $c + q(a')a' = a + ws$,

$$V(s,a) = \max \Big\{ u(ws) + \beta \sum_{s'} \Gamma_{ss'} \bar{V}(s'),$$

$$\max_{c,a'} u[ws + a - q(a')\ a'] + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \Big\}$$

where $\bar{V}(s') = \frac{1}{1-\beta} u(ws')$ is the value of autarky.

- What determines $q(a')$? A zero profit on lenders: Probability of default

Agents in Aiyagari worlds with Extreme Value Shocks

- The fundamental problem

$$v(s,a) = \max_{a',c=sw+aR-a'} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \epsilon(c) + \sum_{s'} \Gamma_{s,s'} \; v(s',a') \right\}$$

- Fix N, a large integer, we approximate the problem by

$$v(s,a) = \max_{a^{n\prime}=sw+aR-c^n,c^n} \left\{ \frac{c^{1-\sigma}-1}{1-\sigma} + \epsilon^n + \sum_{s'} \Gamma_{s,s'} \; v(s',a^{n\prime}) \right\}$$

We have to impute the right probabilities

Overlapping Generations

- Every period there is death and birth of agents.
- We want birth to have new agents be different than existing agents, e.g. poor.
- We want death to prevent certain things such as excessive wealth accumulation.
- We may also want an inefficient economy (the interest rate is too low) and OLG's are natural.
    - May also happen in Aiyagari type economies Aguiar, Amador, and Arellano (2021)
- We may just want to be realistic about the finite nature of the length of life.

- Agents live up to I period
- They own assets A_i,
  - $A_1 = A_{I+1} = 0, \quad \sum_i A_i \mu_i = K$. We may consider different cohort sized μ_i.
- Standard Recursive Representation with State $\{A_2, \cdots, A_i, A_I\}$.
- Many Bells and Whistles are possible.

- Simplest Case, Example Economy.
- $I = 2$, No Storage. Endowment $\{\omega^y, \omega^o\}, \omega^y > \omega^o$.
- $u(c^y, c^o) = \log c^y + \log c^o$
- What happens? Nobody to trade with. So autarky?
- Perhaps there is Money as a store of Value.
- Consider

$$m_t = \frac{\omega^y - c_t^y}{p_t}$$

$$c_{t+1}^o + = \frac{m_t}{p_{t+1} + m_t}$$

What Happens?

- Many Monetary Equilibria $M_t = 1$

- Solutions to a difference equation

$$\frac{\omega^o + \frac{1}{p_{t+1}}}{\omega^y - \frac{1}{p_t}} = \frac{p_{t+1}}{p_t}$$

- A stationary one is $\frac{1}{p^*} = \frac{\omega^y - \omega^o}{2}$.

- There are many more with $P_0 > P^*$, converging to ∞

- Still, Why accept Money from older agents? Who needs them?

Growth Model with Many Firms Suitable for Pandemic Times

- This is a growth model suitable to study business cycles.
- Emphasis on small business creation not on inequality so rep hholds.
- Creation and destruction of small firms both for technological and for financial reasons.
- Household cannot help its small businesses in distress.
- We have in mind that even though Pandemic affects both Supply (want less work) and Demand (Less consumption) there is a reduction in output sold per unit of good produced of $\phi(S)$.

- Two sectors as in Quadrini (2000): Corporate and non corporate sector.
- Corporate sector uses capital and labor via aggr prod fn $F(K, N)$
- Non corporate sector: type/size firms $i \in \{1, \cdots, I\}$, $f^i(n)$, $f^i_n > 0$, (provided the firm has the required number of managers, λ^i).
- A firm requires creation: It costs ξ^i to open a new firm of size i.
- Some Firms are destroyed.
  - Firms invest m in maintenance.
  - Probability that a firm survives is $q^i(m)$, $q^i(0) = 0$, $q^i(\infty) < 1$, $q^i_m > 0$.
- Aggregate measure of type i firms is X_i
- The law of motion of new firms is:

$$X_i' = q^i(M_i)\ X_i + B_i$$

- The Aggregate Feasibility Constraint is

$$C + [K' - (1-\delta)K] + \sum_i X_i\ M_i + \sum_i B_i\ \xi_i = \sum_i X_i\ f_i(N_i) + F(K, N).$$

- Household owns measure x_i of firms of type $i \in \{1, \cdots, \mathcal{I}\}$
- The household may be rationed in its workforce: i.e. it may not be in its static Euler equation.
- Households create b^i new firms of type i at cost ξ^i each,
- Managers choose maintenance and profits.
- In addition to its firms, households own a units of corporate capital which they can increase by savings.
- Households allocate its members to managers, workers or enjoyers of leisure:

$$n + \sum_i \lambda^i x^i + \ell = 1.$$

(implicitly we are guessing (to be verified) that all business are operated).

- Households have preferences over consumption c and leisure ℓ, using utility function $u(c, \ell)$ and discounts the future at rate β.

- Small firms cannot access financing once they are born.
- They can only give benefits to the household:

$$\Omega^i(S) = \max_{n\geq 0, m\leq \psi(S)f^i(n)-w\ n} \psi(S)\ f^i(n) - w\ n - m + \frac{q^i(m)}{R(S')}\ \Omega^i(S')$$

Here, S is the aggregate state and s in the individual state, $\Psi(S) < 1$ is capacity used which is demand determined and $R(S')$ is the rate of return used by the firm.

- Implictly assuming that there is no need to index $\Omega^i(S)$ by s.

Exercise

Get the FOC assuming first that m is unrestricted and then that $m \leq \psi(S)f^i(n) - w\ n$.

$$V(S, a, x_1, \cdots, x_I) = \max_{c,n,b_1,\cdots,b_I,a'} u(c,\ 1 - n - \sum_i \lambda^i\ x^i) + \beta\ V(S', a', x_1', \cdots, x_I') \qquad s.t.$$

$$c + \sum_i\ b_i\ \xi_i + a' = n\ w(S) + a\ R(S) + \sum_i\ \pi_i(S)\ x_i$$

$$x_i' = q^i(M_i)\ x_i + b_i \qquad i \in \{1, \cdots, I\}.$$

Exercise

Get the FOCs for b^i a' and n assuming first that $\lambda^i = 0$ and $\pi^i > 0$ and charaterize the solution (the relation between the FOC of b^i, m^i and a'). Then characterize the FOC when $\lambda^i > 0$.

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