Course in Heterogeneity: Econ 081

V: Banking in General Equilibrium

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A Model with Banks, Search & Matching, Putty Clay and Working Capital

- Putty Clay with Decreasing Returns to scale firms.
- Exogenous firm destruction Rate δ_1, depreciation δ_2, and workers quits δ_3.
- Firms need to prepay some intermediate inputs and labor
- Timing:
    - Prepay workers. All interest disappears.
    - Buy intermediate goods one period in advance
- Firms start with long term and working capital loans from banks.
- Workers need to be hired via vacancy posting/training costs c^v
- A Calvo fairy allows firms to be sold to mutual funds.
- Wages set up in a variety of ways.
- Households like deposits

A representative bank

- Issues deposits and bonds. Equity issuance is very costly.
- Makes long term loans to new firms and short term loans to all firms as working capital.
- Incurs administration costs to generate assets.
- Strong appetite for borrowing is disciplined by the capital regulation.

Steady State

Equilibrium Objects: Prices and Profits

- Liquid asset saving rate: Bonds and Mutual Fund: q.
- Working capital $q^w > q$.
- Deposits $q^d > q > q^w$.
- Loans $1 + r^\ell = 1/q^\ell$
- Wages (we will worry about them later): w.
- Profits of π^f and π^B
- Government transfers T
- New hires N^n

Households care for but don't choose labor, get paid late

Take as given $\{q, q^w, q^d, w, \pi^f, \pi^B, T, N\}$

$$V(a,n) = \max_{c,b,d} \; u(c,n) + \nu(d) + \beta V(a',n') \qquad \text{s.t.}$$

$$c + q\ b + q^d\ d = a + w\ n + \pi^f + \pi^b + T$$
$$a' = b + d$$

FOCs are

$$q = \beta$$
$$q^d = \frac{\nu_d(D)}{u_c(C,N)} + q$$

For the time being, households supply the labor demanded by firms.

Households decisions in SS

$$q = \beta \tag{1}$$

$$q^d = \frac{\nu_d(D)}{u_c(C, N)} + q \tag{2}$$

Firms: Decreasing returns to scale with $\{k, m, n\}$

- Maybe Cobb-Douglas: $y = z\ k^{\alpha} n^{\gamma} m^{\theta}, \quad \alpha + \gamma + \theta < 1.$

- Firms must pay both wage bills and intermediate goods one period before production. No negative dividends.

- Borrowing is expensive, and firms liquid assets are in the form of bonds. This means that the discount rates $q(h')$ on liquid assets that they face is

$$q(h') = \begin{cases} q(1-\delta_1) & \text{if } h' < 0, \\ (q^w, q(1-\delta_1)) & \text{if } h' = 0, \\ q^w & \text{if } h > 0, \end{cases} \qquad\qquad q^w < q(1-\delta_1).$$

- This means that when $h' = 0$, the effective rate of return used in choosing inputs is in between q^w and $q(1-\delta_1)$.

Equity Firms Solve ($c^v = 0$)

- Equity firms have already been turned to the mutual fund

$$\Omega(k, m, n, h) = \max_{n', m', h'} zk^{\alpha} n^{\gamma} m^{\theta} - h + h' \; q(h') - w \; n' \\ - m' - \delta_2 k + (1 - \delta_1') \; q \; \Omega(k, m', n', h')$$

$$\text{s.t.} \qquad 0 \leq z \; k^{\alpha} n^{\gamma} m^{\theta} - h + h' q(h') - m' - wn' - \delta_2 k.$$

- Dividends are

$$div = zk^{\alpha} n^{\gamma} m^{\theta} - h + h' q(h') - m' - wn' - \delta_2 k$$

MUTUAL FUND BUYS LOAN FIRMS WITH PROB λ $(c^v = 0)$

- Loan firms switch to equity firms with probability λ after making all their decisions.

- The mutual fund buys those switching firms by paying $k + \kappa^f$ before failure shocks δ_1 happens.

- Under the SS prices, the corporate debt is increasing in age due to the interest payment on $k + \kappa^f$. The mutual fund pays any outstaning balance of workig capital loans in excess of $\frac{m' + w\ n'}{q^W}$ for surviving firms before they start operating as equity firms. For the time being, we focus on the case where $c^v = 0$.

Loan firms: switch to equity firms with prob λ $\quad (c^v = 0)$

$$\Pi(k, m, n, h) = \max_{n', m', h'} z\ k^{\alpha}\ n^{\gamma}\ m^{\theta} - h + h'\ q(h') - m' - w\ n' - r^{\ell}(k + \kappa^{f}) - \delta_{\mathbf{2}} k$$

$$+ (1 - \delta_{\mathbf{1}}')(1 - \lambda)\ q\ \Pi(k, m', n', h')$$

$$+ (1 - \delta_{\mathbf{1}}')\ \lambda\ q\ \left[\Omega\left(k, m', n', \frac{m' + wn'}{q^{w}}\right) - \left(h' - \frac{m' + wn'}{q^{w}}\right)\right]$$

$$\underbrace{-\lambda\left(k + \kappa^{f}\right)}_{\textbf{paid by the mutual fund}} \quad . \tag{3}$$

[↑To be consistent with the bank's problem, the maturity shock happens before $\delta_{\mathbf{1}}$.]

subject to

$$0 \leq z\ k^{\alpha}\ n^{\gamma}\ m^{\theta} - h + h'\ q(h') - m' - w\ n' - r^{\ell}(k + \kappa^{f}) - \delta_{\mathbf{2}} k$$

Some static decisions of loan and equity firms $(c^v = 0)$

1. Borrowing firms with high debt: solve FOC with q^w as a function of k.

$$n^w = \left[q^w \ z \ k^\alpha \left(\frac{\gamma}{w}\right)^{1-\theta} \theta^\theta\right]^{\frac{1}{1-\theta-\gamma}}, \quad m^w = \left[q^w \ z \ \theta \ k^\alpha \ (n^w)^\gamma\right]^{\frac{1}{1-\theta}}$$

2. Unconstrained firms with low debt: solve FOC with q as a function of k.

$$\widetilde{n} = \left[q(1-\delta_1) \ z \ k^\alpha \left(\frac{\gamma}{w}\right)^{1-\theta} \theta^\theta\right]^{\frac{1}{1-\theta-\gamma}}, \quad \widetilde{m} = \left[q(1-\delta_1) \ z \ \theta \ k^\alpha \ (\widetilde{n})^\gamma\right]^{\frac{1}{1-\theta}}.$$

3. Limbo firms with an intermediate level of debt: for a shadow price $\widehat{q}(m,n,h)$,

$$\widehat{n}(m,n,h) = \left[\widehat{q} \ z \ k^\alpha \left(\frac{\gamma}{w}\right)^{1-\theta} \theta^\theta\right]^{\frac{1}{1-\theta-\gamma}}, \quad \widehat{m}(m,n,h) = \left[\widehat{q} \ z \ \theta \ k^\alpha \ (\widetilde{n})^\gamma\right]^{\frac{1}{1-\theta}}.$$

$$0 = z \ k^\alpha \ n^\gamma \ m^\theta - h - \widehat{m}(m,n,h) - w \ \widehat{n}(m,n,h) - r^\ell(k + \kappa^f) - \delta_2 k \text{ (Loan firms)}$$

$$0 = z \ k^\alpha \ n^\gamma \ m^\theta - h - \widehat{m}(m,n,h) - w \ \widehat{n}(m,n,h) - \delta_2 k \text{ (Equity firms)}$$

[↑ For $c^v > 0$, the static decision rules hold if the mutual fund pays corporate debt in excess of $\frac{m' + (w + c^v)n'}{q^W}$ rather than $\frac{m' + (w + c^v)n' - c^v(1-\delta_3)n}{q^W}$.]

Algorithm to Solve Loan Firms' Problem: $\{k, w, q, q^w\}$ given and $c^v = 0$

Since the equilibrium prices in SS imply $h_j^\ell > h_{j-1}^\ell$, loan firms always borrow at q^w:

4. Compute $h_0^\ell = \frac{m^w + wn^w}{q^w}$.

5. For $j = 1, 2, \ldots,$

$$
\begin{aligned}
m_j^\ell &= m^w \\
n_j^\ell &= n^w \\
h_j^\ell &= -\frac{1}{q^w}\Big\{ zk^\alpha (m^w)^\theta (n^w)^\gamma - r^\ell (k + \kappa^f) - \delta_2 k - h_{j-1}^\ell - (m^w + w\ n^w)\Big\}
\end{aligned}
$$

Algorithm to Solve Equity Firm Problem I: $\{k, w, q, q^w\}$ given and $c^v = 0$

6. We already have solved FOC for $\{n^w, m^w\}$ with q^w and $\{\widetilde{n}, \widetilde{m}\}$ with q as a function of k.
7. Regardless of the switching history, equity firms start from the same condition. Thus, we need to solve the equity firm's problem once and for all with $h_{\mathbf{0}}^e = \frac{m^w + wn^w}{q^w}$, $m_{\mathbf{0}}^e = m^w$, and $n_{\mathbf{0}}^e = n^w$. Use $h_{\mathbf{0}}^e$ to evaluate the initial condition of the new equity firm:
   7.1 If $0 < zk^\alpha (m^w)^\theta (n^w)^\gamma - \frac{m^w + wn^w}{q^w} - \delta_2 k - (\widetilde{m} + w\ \widetilde{n})$,
   goto 8.
   7.2 else if $0 > zk^\alpha (m^w)^\theta (n^w)^\gamma - \frac{m^w + wn^w}{q^w} - \delta_2 k - (m^w + w\ n^w)$,
   goto 9.
   7.3 else
   goto 10.
8. Start from unconstrained equity firms: For $t = 1, \ldots, T$

$$
\begin{aligned}
h_t^e &= 0 \\
m_t^e &= \widetilde{m} \\
n_t^e &= \widetilde{n} \\
div_t &= zk^\alpha (m_{t-\mathbf{1}}^e)^\theta (n_{t-\mathbf{1}}^e)^\gamma - h_{t-\mathbf{1}}^e - \delta_{\mathbf{2}} k - (\widetilde{m} + w\ \widetilde{n})
\end{aligned}
$$

Algorithm to Solve Equity Firm Problem II: $\{k, w, q, q^w\}$ given and $c^v = 0$

9. Start from constrained equity firms: For $t = 1, \dots, T$,

9.1 Constrained:

$$m_t^e = m^w$$

$$n_t^e = n^w$$

$$h_t^e = \frac{1}{q^w}\Big\{ zk^\alpha (m_{t\text{-}1}^e)^\theta (n_{t\text{-}1}^e)^\gamma - \delta_2 k - h_{t\text{-}1}^e - [m^w + w\ n^w]\Big\}.$$

$$\text{if}\begin{cases} h_t^e > 0 & \text{continue,} \\ \text{else} & \text{let } \widetilde{t}_1 = t \text{ and goto 9.2.}\end{cases}$$

9.2 Investing all profits. For $t \geq \widetilde{t}_1$, $h_t^e = 0$. $m_t^e, n_t^e, \widehat{q}_t$ satisfy the following conditions:

$$n_t^e = \widehat{n}(m_{t\text{-}1}^e, n_{t\text{-}1}^e, h_{t\text{-}1}^e; \widehat{q}_t)$$

$$m_t^e = \widehat{m}(m_{t\text{-}1}^e, n_{t\text{-}1}^e, h_{t\text{-}1}^e; \widehat{q}_t)$$

$$0 = zk^\alpha (m_{t\text{-}1}^e)^\theta (n_{t\text{-}1}^e)^\gamma - h_{t\text{-}1}^e - \delta_2 k - [m_t^e + w\ n_t^e]$$

$$\text{if}\begin{cases} \widehat{q}_t < (1-\delta_1) q & \text{continue,} \\ \text{else} & \text{let } \widetilde{t}_2 = t \text{ and goto 9.3.}\end{cases}$$

Algorithm to Solve Equity Firm Problem III: $\{k, w, q, q^w\}$ given and $c^v = 0$

10. Start from limbo equity firms: $\tilde{t}_1 = 1$. For $t = 1, \ldots, T$,

10.1 Investing all profits. $h_t^e = 0$,

$$n_t^e = \widehat{n}(m_{t\text{-}1}^e, n_{t\text{-}1}^e, h_{t\text{-}1}^e; \widehat{q}_t)$$

$$m_t^e = \widehat{m}(m_{t\text{-}1}^e, n_{t\text{-}1}^e, h_{t\text{-}1}^e; \widehat{q}_t)$$

$$0 = zk^{\alpha}(m_{t\text{-}1}^e)^{\theta}(n_{t\text{-}1}^e)^{\gamma} - h_{t\text{-}1}^e - \delta_2 k - [m_t^e + wn_t^e]$$

$$\text{if} \begin{cases} \widehat{q}_t < (1-\delta_1)q & \text{continue,} \\ \text{else} & \text{let } \tilde{t}_2 = t \text{ and goto 10.2.} \end{cases}$$

10.2 Paying dividends: For $t \geq \tilde{t}_2$, $h_t^e = 0, m_t^e = \tilde{m}, n_t^e = \tilde{m}$, and

$$div_t = zk^{\alpha}(m_{t\text{-}1}^e)^{\theta}(n_{t\text{-}1}^e)^{\gamma} - h_{t\text{-}1}^e - \delta_2 k - [m_t^e + w\ n_t^e]$$

ENTRANTS ($c^v = 0$)

- A prospective enterant solves for

$$k^* = \arg\max_k \left\{ q\ (1 - \delta_1)\ \Pi\left(k, m_0^\ell, n_0^\ell, \frac{m_0^\ell + wn_0^\ell}{q^w}\right)\right\} \tag{4}$$

- Note that n_0^ℓ and m_0^ℓ are functions of k.

- In equilibrium, the free-entry condition must be satisfied:

$$0 = \Pi\left(k^*, m_0^\ell, n_0^\ell, \frac{m_0^\ell + w\ \ n_0^\ell}{q^w}\right) \tag{5}$$

Evaluating the value of entry I $(c^v = 0, \Delta h \geq 0)$

- The value of loan firms is given by

$$\Pi_j = (1-\delta_1)(1-\lambda)q\Pi_{j+1} + (1-\delta_1)\lambda q \left[\Omega_{j+1,j+1} - \frac{k + \kappa^f}{q(1-\delta_1)} - \left(h^\ell_j - \frac{m^w + wn^w}{q^w}\right)\right]$$

<u>Debt in age j</u>

- Let $c^w(k) = zk^\alpha(m^w)^\theta(n^w)^\gamma - r^\ell(k+\kappa^f) - \delta_2 k - (m^w + wn^w)$. c^w is a cash flow of loan firms before repaying their short-term debt. This is constant over age given (w, k). Because of the interest payment on $k + \kappa^f$, h^ℓ_j grows over time at a constant rate.

$$\begin{aligned}
h^\ell_j &= \frac{1}{q^w}\left[h^\ell_{j-1} - c^w(k)\right] \\
&= \left(\frac{1}{q^w}\right)^j h^\ell_0 - \sum_{s=1}^{j}\left(\frac{1}{q^w}\right)^s c^w(k) \\
&= \left(\frac{1}{q^w}\right)^j \left\{h^\ell_0 - \frac{c^w(k)}{1-q^w}\right\} + \frac{c^w(k)}{1-q^w}
\end{aligned}$$

Evaluating the value of entry II ($c^v = 0, \Delta h \geq 0$)

- **The value of a new equity firm**

  Because every equity firm starts with the same amount of debt and working/physical capital, $\Omega_{j,j}$ is identical for all j: $\Omega_{j,j} = \overline{\Omega}$.

- A new equity firm may or may not be paying out dividends upon switching. By solving the equity firm's decision rules, we know the period $\widetilde{t}_2$ in which dividend payments start after switching. Using this information and decisions in periods $\widetilde{t}_2$ - 1 and $\widetilde{t}_2$,

$$\begin{aligned}\overline{\Omega} &= [q(1-\delta_1)]^{\widetilde{t}_2 - 1}\,\Omega_{\widetilde{t}_2} \\ &= [q(1-\delta_1)]^{\widetilde{t}_2 - 1}\left[zk^{\alpha}(n_{\widetilde{t}_2 - 1})^{\gamma}(m_{\widetilde{t}_2 - 1})^{\theta} - h_{\widetilde{t}_2 - 1} - \delta_2 k - \widetilde{m} - w\widetilde{n} + q(1-\delta_1)\widetilde{\Omega}\right],\end{aligned}$$

where

$$\widetilde{\Omega} = \frac{zk^{\alpha}(\widetilde{n})^{\gamma}(\widetilde{m})^{\theta} - \delta_2 k - \widetilde{m} - w\widetilde{n}}{1 - q(1-\delta_1)}.$$

is the value of an equity firm with $(n, m, h) = (\widetilde{n}, \widetilde{m}, 0)$.

Evaluating the value of entry III ($c^v = 0, \Delta h \geq 0$)

- **The value of entry**

$$
\begin{aligned}
\Pi_1 &= q(1-\delta_1)(1-\lambda)\Pi_2 + q(1-\delta_1)\lambda\left[\overline{\Omega} + \frac{m^w + wn^w}{q^w} - h_1^\ell - \frac{k + \kappa^f}{q(1-\delta_1)}\right] \\
&= \frac{q(1-\delta_1)\lambda}{1 - q(1-\delta_1)(1-\lambda)}\left[\overline{\Omega} + \frac{m^w + wn^w}{q^w} - \frac{c^w(k)}{1-q^w} - \frac{k + \kappa^f}{q(1-\delta_1)}\right] - \frac{q(1-\delta_1)\lambda}{q^w - q(1-\delta_1)(1-\lambda)}\left\{h_0^\ell - \frac{c^w(k)}{1-q^w}\right\}
\end{aligned}
$$

- **For the equity firm's debt to shrink over time**

$$
h' - h > 0 \iff \underbrace{\frac{m^w + wn^w}{q^w}}_{\text{initial debt}} < \underbrace{\frac{zk^\alpha (n^w)^\gamma (m^w)^\theta - m^w - wn^w - \delta_2 k}{1-q^w}}_{\text{PV of all cash flow}}
$$

To ensure that this condition is met for a given w, the range of k in which the optimal k^* is searched for is

$$
k \in \left(0, \left\{\frac{(1-\gamma-\theta)\left[zq^w\left(\frac{\gamma}{w}\right)^\gamma \theta^\theta\right]^{\frac{1}{1-\gamma-\theta}}}{q^w\delta_2}\right\}^{\frac{1-\gamma-\theta}{1-\alpha-\gamma-\theta}}\right).
$$

Age and the measure of firms:

- Age j indicates the passage of time since a firm is created.
- x_j^ℓ is the measure of loan firms of age j.
- $X_{i,j}^e$ is the measure of age-j firms that have been equity firms since age i.

$$x_j^\ell = (1-\delta_1)^j \, (1-\lambda)^{j-1} \, x_0^\ell,$$

$$x_{ij}^e = (1-\delta_1)^j \, (1-\lambda)^{i-2} \, \lambda \, x_0^\ell$$

$$\frac{\#(\text{loan firms})}{\#(\text{equity firms})} = \frac{\delta_1}{(1-\delta_1)\lambda}$$

Example: Loan firms

age(j)	states	production	decisions	measure
0	no states	no production	$k, n_0^\ell, m_0^\ell, h_0^\ell$	x_0^ℓ
1	$k, n_0^\ell, m_0^\ell, h_0^\ell$	$k^\alpha (n_0^\ell)^\gamma (m_0^\ell)^\theta$	$n_1^\ell, m_1^\ell, h_1^\ell$	$x_1^\ell = (1-\delta_1)^1 (1-\lambda)^0 x_0^\ell$
2	$k, n_1^\ell, m_1^\ell, h_1^\ell$	$k^\alpha (n_1^\ell)^\gamma (m_1^\ell)^\theta$	$n_2^\ell, m_2^\ell, h_2^\ell$	$x_2^\ell = (1-\delta_1)^2 (1-\lambda)^1 x_0^\ell$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	$k, n_{t-1}^\ell, m_{t-1}^\ell, h_{t-1}^\ell$	$k^\alpha (n_{t-1}^\ell)^\gamma (m_{t-1}^\ell)^\theta$	$n_t^\ell, m_t^\ell, h_t^\ell$	$x_t^\ell = (1-\delta_1)^t (1-\lambda)^{t-1} x_0^\ell$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Aggregation: definition I ($c^v = 0$)

- **GDP**:

$$Y = \Upsilon^f - \underbrace{\left(\widehat{M}^f + m_0^\ell x_0^\ell\right)}_{\textbf{intermediate goods}} - \underbrace{\delta_1 w \left(\widehat{N}^f + x_0^\ell n_0^\ell\right)}_{\textbf{default cost}} - \underbrace{\left(\gamma^d D + \psi(\ell^n, h^b)\right)}_{\textbf{banking costs}}$$

- **Aggregate output**:Let $y_j^\ell = k^{*\alpha}(m_j^\ell)^\theta \, (n_j^\ell)^\gamma$ and $y_{i,j}^e = k^{*\alpha}(m_{i,j}^e)^\theta \, (n_{i,j}^e)^\gamma$.

$$\Upsilon^f = \sum_{j=1}^{\infty} x_j^\ell y_{j-1}^\ell + \sum_{i=2}^{\infty}\sum_{j=i}^{\infty} x_{i,j}^e \, y_{i,j-1}^e$$

- **Aggregate demand for intermediate goods by existing firms**:

$$\widehat{M}^f = \sum_{j=1}^{\infty} x_j^\ell \, m_j^\ell + \sum_{i=2}^{\infty}\sum_{j=i}^{\infty} x_{i,j}^e \, m_{i,j}^e$$

- **Aggregate (beginning-of-period) employment and vacancy postings by existing firms**:

$$N^f = \sum_{j=1}^{\infty} x_j^\ell \, n_{j-1}^\ell + \sum_{i=2}^{\infty}\sum_{j=i}^{\infty} x_{i,j}^e \, n_{i,j-1}^e, \quad \widehat{N}^f = \sum_{j=1}^{\infty} x_j^\ell \, n_j^\ell + \sum_{i=2}^{\infty}\sum_{j=i}^{\infty} x_{i,j}^e \, n_{i,j}^e$$

The relationship between N^f and $\widehat{N}^f$ is

$$N^f = (1-\delta_1)\left(\widehat{N}^f + x_0^\ell n_0^\ell\right)$$

Aggregation: definition II

- **Aggregate physical capital stock and aggregate investment**:

$$K^f = k^* \left\{ \sum_{j=1}^{\infty} x_j^{\ell} + \sum_{i=2}^{\infty} \sum_{i=j}^{\infty} x_{i,j}^e \right\}, \quad I^f = (k^* + \kappa^f) x_{\mathbf{0}}^{\ell} + \delta_{\mathbf{2}} K^f$$

- **Aggregate working capital loan demand and (beginning-of-the-period) repayment**:

$$\widehat{H}^{f+} = q^w \left[\sum_{j=1}^{\tau_{\mathbf{1}}-\mathbf{1}} x_j^{\ell} \; h_j^{\ell} + \sum_{i=2}^{\infty} \sum_{j=i}^{\tilde{\tau}_{\mathbf{1}}^i-\mathbf{1}} x_{i,j}^e \; h_{i,j}^e \right], \quad H^{f+} = \sum_{j=1}^{\tau_{\mathbf{1}}} x_j^{\ell} \; h_{j-\mathbf{1}}^{\ell} + \sum_{i=2}^{\infty} \sum_{j=i}^{\tilde{\tau}_{\mathbf{1}}^i} x_{i,j}^e \; h_{i,j-\mathbf{1}}^e$$

The relationship between $\widehat{H}^f$ and H^f is

$$H^{f+} = (1 - \delta_{\mathbf{1}}) \left(\frac{\widehat{H}^{f+}}{q^w} + x_{\mathbf{0}}^{\ell} h_{\mathbf{0}}^{\ell} \right)$$

- **Aggregate net firm profits**:

$$\pi^f = \sum_{i=2}^{\infty} \sum_{j=\tilde{\tau}_{\mathbf{2}}^i}^{\infty} x_{i,j}^e divs_{i,j} - (k^* + \kappa^f) \sum_{j=2}^{\infty} \frac{x_{j,j}^e}{1 - \delta_{\mathbf{1}}} - \sum_{j=2}^{\infty} x_{j,j}^e \left(h_{j-\mathbf{1}}^{\ell} - h_{\mathbf{0}}^{\ell} \right)$$

Aggregation when $\Delta h \geq 0$

Let $\widetilde{t}_2$ be the period when equity firms start paying out dividends after switching from loan firms.

$$\Upsilon^f = \frac{(1-\delta_1)x_0^\ell}{1-(1-\delta_1)(1-\lambda)} y^w + \frac{(1-\delta_1)^2 \lambda x_0^\ell}{1-(1-\delta_1)(1-\lambda)} \sum_{t=0}^{\widetilde{t}_2 - 1} (1-\delta_1)^t y_t^e + \frac{(1-\delta_1)^{\widetilde{t}_2+2} \lambda x_0^\ell}{1-(1-\delta_1)(1-\lambda)} \frac{\widetilde{y}}{\delta_1}$$

- To compute N^f, replace $(y^w, y_t^e, \widetilde{y})$ with $(n^w, n_t^e, \widetilde{n})$.
- To compute $\widehat{N}^f$, replace $(y^w, y_t^e, \widetilde{y})$ with $(n^w, n_{t+1}^e, \widetilde{n})$.
- To compute $\widehat{M}^f$, replace $(y^w, y_t^e, \widetilde{y})$ with $(m^w, m_t^e, \widetilde{m})$.

$$K^f = k^* \frac{(1-\delta_1)x_0^\ell}{\delta_1}$$

$$\widehat{H}^{f+} = q^w \left\{ \frac{(1-\delta_1)x_0^\ell}{q^w - (1-\delta_1)(1-\lambda)} \left[h_0^\ell - \frac{c^w(k)}{1-q^w} \right] + \frac{(1-\delta_1)x_0^\ell}{1-(1-\delta_1)(1-\lambda)} \frac{c^w(k)}{1-q^w} + \frac{(1-\delta_1)^2 \lambda x_0^\ell}{1-(1-\delta_1)(1-\lambda)} \sum_{t=0}^{\widetilde{t}_1 - 2} (1-\delta_1)^t h_{t+1}^e \right\}$$

$$\pi^f = \frac{(1-\delta_1)^{\widetilde{t}_2+1} \lambda x_0^\ell}{1-(1-\delta_1)(1-\lambda)} \left[div_{\widetilde{t}_2} + \frac{(1-\delta_1)}{\delta_1} \widetilde{div} \right] - (k^* + \kappa^f) \frac{(1-\delta_1)\lambda x_0^\ell}{1-(1-\delta_1)(1-\lambda)}$$
$$- \left[\frac{1}{q^w - (1-\delta_1)(1-\lambda)} - \frac{1}{1-(1-\delta_1)(1-\lambda)} \right] (1-\delta_1)^2 \lambda \left[h_0 - \frac{c^w}{1-q^w} \right] x_0^\ell$$

Banks: $\{r^{\ell}, q^{d}, q^{w}, q\}$ given

$$V\left(a^{b}, \ell^{b}\right) = \max_{c^{b}, h^{b}, \ell^{n}, b^{h}, D^{b}} c^{b} + \beta^{b} \, V\left(a^{b'}, \ell^{b'}\right) \qquad \text{s.t.}$$

$$(TL) \qquad \ell^{b'} = (1-\delta_{\mathbf{1}}') \left[(1-\lambda)\, \ell^{b} + \ell^{n}\right] \quad \leftarrow \text{entrants can die}$$

$$(TA) \qquad a' = \left[\lambda + r^{\ell}\right] \ell^{b'} + (1-\delta_{\mathbf{1}}')\, h^{b} - b^{h} - D^{b}$$

$$(BC) \qquad c^{b} + \chi(c^{b}) + \ell^{n} + \psi(\ell^{n}, h^{b}) + q^{w}\; h^{b} + \gamma^{d} D^{b} + \phi\,(\varsigma) \left[(1-\lambda)\ell^{b} + \ell^{n} + q^{w} h^{b}\right] + \iota \leq a + q\; b^{h} + q^{d} D^{b}$$

$$(EQ) \qquad e^{b} = (1-\lambda)\ell^{b} + \ell^{n} + q^{w} h^{b} - q\; b^{b} - q^{d} D^{b}$$

$$(KR) \qquad \varsigma = \frac{e^{b}}{\omega_{\mathbf{1}}[(1-\lambda)\ell^{b} + \ell^{n}] + \omega_{\mathbf{2}}\; q^{w}\; h^{b}}$$

- ι is a financial shock (zero in SS).
- Banks probl yield choices: $\{\ell^{b}, \ell^{n}, h^{b}, b^{b}, D^{b}, c^{b}, e^{b}, \varsigma\}$

Optimal conditions for banks in SS

$$[\ell^n] \qquad \psi_n(\ell^n, h^b) + 1 + \phi(\varsigma) - \frac{q-\beta^b}{q}(1-\omega_1\varsigma) = \beta^b(1-\delta_1)\left(1+r^\ell\right) \tag{6}$$

$$[h^b] \qquad \psi_h(\ell^n, h^b) + q^w + \phi(\varsigma)q^w - \frac{q-\beta^b}{q}q^w(1-\omega_2\varsigma) = \beta^b(1-\delta_1) \tag{7}$$

$$[b] \qquad \frac{q-\beta^b}{q}\left[\omega_1\frac{1}{1-(1-\delta_1)(1-\lambda)}\ell^n + \omega_2 q^w h^b\right] = -\phi_\varsigma(\varsigma)\left[\frac{1}{1-(1-\delta_1)(1-\lambda)}\ell^n + q^w h^b\right] \tag{8}$$

$$[D] \qquad q^d = q\left(1+\frac{\gamma^d}{\beta^b}\right) \tag{9}$$

<u>CASE A</u>: In the special case where $\psi = 0$ and $\omega_1 = \omega_2 = \omega$, these conditions pin down r^ℓ, q^w, q^d and ς, given q.

$$[b] \qquad -\phi_\varsigma(\varsigma) = \frac{q-\beta^b}{q}\omega \tag{10}$$

$$[\ell^n] \qquad 1+r^\ell = \frac{1+\phi(\varsigma) - \frac{q-\beta^b}{q}(1-\omega\varsigma)}{\beta^b(1-\delta_1)} \tag{11}$$

$$[h^b] \qquad q^w = \frac{1}{1+r^\ell} \tag{12}$$

$$[D] \qquad q^d = q\left(1+\frac{\gamma^d}{\beta^b}\right) \tag{13}$$

Government

- Collects $\phi\left(\varsigma\right)\left[(1-\lambda)\ell^{b}+\ell^{n}+q^{w}h^{b'}\right]$

- Gives it back to the households as $T=\phi(\varsigma)\left[(1-\lambda)\ell^{b}+\ell^{n}+q^{w}h^{b'}\right]$

$[k^*]$ *eqn* (4)   Optimal k of entrants

$[x_0^\ell]$ *eqn* (5)   Zero Profit of entrants

$$[q^d] \qquad D = D^h \tag{14}$$

$$[q^w] \qquad (1-\delta_1)H^b = H^{f+} \tag{15}$$

$$[w] \qquad N^f = .8 \tag{16}$$

$$[D] \qquad B^b = B^h \tag{17}$$

$$[r^\ell] \qquad \ell^n = \left[k^* + \kappa^f\right] x_0^\ell \tag{18}$$

$$[q] \qquad Y = C^h + I^f \tag{19}$$

$$[T] \qquad T = \phi\left(\varsigma^b\right)\left[(1-\lambda)\ell^b + \ell^n + q^w h^b\right] \tag{20}$$

Algorithms to solve the Problem in SS

- Household's problem implies that $q = \beta$.

- Depending on the nature of the bank's parameters, the remaining prices are restricted in different ways. In general, we have to solve for $\{q^w, r^\ell, w, k, x_0\}$, but there are cases where it is simpler.

  - **Case A** ($\psi = 0, \omega_1 = \omega_2$): Aggregation given w and x_0^ℓ.
    The problem of the bank determines q^w, r^ℓ, q^d and ς by itself. The firm's problem yields k so the main routine only needs to solve for $\{w, x_0\}$.

Case A ($\psi = 0, \omega_1 = \omega_2$): Aggregation given w and x_0^ℓ

1. Given q, the problem of the bank determines q^d, q^w, r^ℓ and ς by itself.
2. Given $\{q, q^d, q^w, r^\ell, \varsigma\}$, the firm's problem yields $\{k, (n_j^\ell, m_j^\ell, h_j^\ell)_{j=0}^\infty,$ $(n_{i,j}^e, m_{i,j}^e, h_{i,j}^e)_{i=2,\ j\geq i}^\infty\}$
3. Given previous and x_0, aggregation yields N^f, $\widehat{M}^f$, K^f, I^f, π^f, ℓ^n, H^{f+}, Υ^f.
4. Given previous, the hhold problem yields C^h, D^h and Y.
5. The definition of ς yields B^b.

- The main routine only needs to solve for $\{w, x_0\}$.
- It uses two equations that can be solved for independently

$$\begin{aligned} N(x_0) &= 0.8, && (21) \\ \Pi(w) &= 0.0. && (22) \end{aligned}$$

-

Case A: Bank's problem

- INPUT OF THE PROBLEM: q
- OUTPUT OF THE PROBLEM q^d,q^w, r^ℓ and ζ
- Indeterminate: L, B, D

Case A: $\omega = \omega_1 = \omega_2$ Bank's decisions

- ς is the capital ratio of bank and ω is the risk weight. Bank's optimal conditions are summarized as below:

$$[b] \qquad -\phi_\varsigma(\varsigma) = \frac{q - \beta^b}{q}\omega \tag{23}$$

$$[\ell^n] \qquad 1 + r^\ell = \frac{1 + \phi(\varsigma) - \frac{q-\beta^b}{q}(1 - \omega\varsigma)}{\beta^b(1 - \delta_1)} \tag{24}$$

$$[h^b] \qquad q^w = \frac{1}{1 + r^\ell} \tag{25}$$

$$[D] \qquad q^d = q\left(1 + \frac{\gamma^d}{\beta^b}\right) \tag{26}$$

- There is also a budget constraint that determines bank dividend payouts, which can be backed out once the equilibrium quantities are determined.

Case A: Firms' decisions and Aggregation

- INPUT OF THE PROBLEM: $\{q, q^w, r^\ell, w, x_0\}$

- OUTPUT OF THE PROBLEM: $\{k, (n_j^\ell, m_j^\ell, h_j^\ell), (n_{i,j}^e, m_{i,j}^e, h_{i,j}^e)\}$ and N^f, $\widehat{M}^f$, K^f, I^f, π^f, ℓ^n, H^{f+}

Case A: Firms' decisions given $\{q, q^w, r^\ell, w\}$

A. The life-cycle decisions of firms:

1. Given $\{q, q^w, r^\ell, w\}$, solve for the optimal firm size k^*. For each k,

   - loan firms: (n^w, m^w, h_j^ℓ) for age $j = 1, \dots$
   - equity firms: $(n^e_{i,j}, m^e_{i,j}, h^e_{i,j})$ for age $j = 1, \dots,\ i \leq j$
   - entry value: $\Pi(k, q, q^w, r^\ell, w)$

B. Optimal capital: $k^* = \arg\max_k \Pi(k, q, q^w, r^\ell, w)$

Case A: Aggregation given x_0^ℓ and firms' choice of k, n, m, h

C. Aggregation of decisions: given the firms' decisions, k^* and $x_{\mathbf{0}}^\ell$, we get

- Υ^f: aggregate output (beginning-of-period)
- N^f: aggregate employment (beginning-of-period)
- $\widehat{N}^f$: aggregate employment for next period chosen by existing firms
- $\widehat{M}^f$: aggregate demand for intermediate goods for next period chosen by existing firms
- K^f: aggregate physical capital (beginning-of-period)
- $I^f = (k^* + \kappa^f)x_{\mathbf{0}}^\ell + \delta_{\mathbf{2}} K^f$
- $\widehat{H}^{f+}$: aggregate working capital loan demanded by existing firms to finance next period's production costs
- π^f: aggregate corporate profits
- $\ell^n = \left(k^* + \kappa^f\right) x_{\mathbf{0}}^\ell$
- $H^b = (\widehat{H}^{f+}/q^w + h_{\mathbf{0}}^\ell x_{\mathbf{0}}^\ell)$

Case A: Household's decisions and GDP given $\{\Upsilon^f, N^f, I^f, \widehat{N}^f, \widehat{M}^f, m_0^\ell, n_0^\ell, x_0^\ell, H^b, \ell^n, q^w, q, q^d, \varsigma\}$

$$q^d = \frac{\nu_d(D^h)}{u_c\left(C^h, N^f\right)} + q \tag{27}$$

$$Y = \Upsilon^f - \left(\widehat{M}^f + m_{\mathbf{0}}^\ell x_{\mathbf{0}}^\ell\right) - \delta_{\mathbf{1}} w \left(\widehat{N}^f + n_{\mathbf{0}}^\ell x_{\mathbf{0}}^\ell\right) - \gamma^d D^b \tag{28}$$

$$Y = C^h + I^f \tag{29}$$

- (27)-(29) determine $D^h (= D^b)$, Y and C^h.
- $B^h = B^b$ is backed out from the definition of capital ratio:

$$\varsigma = \frac{(1-\lambda)\ell^b(\ell^n) + \ell^n + q^w H^b - qB^b - q^d D^b}{\omega_{\mathbf{1}}\left[(1-\lambda)\ell^b(\ell^n) + \ell^n\right] + \omega_{\mathbf{2}} q^w H^b}$$

Functional forms

Utility function:

$$u(c) + v(d) = \frac{c^{1-\sigma^c}}{1-\sigma^c} + \nu^d \frac{d^{1-\sigma^d}}{1-\sigma^d}$$

Capital regulation:

$$\phi(\varsigma) = \eta_0 \exp\left(\frac{\theta^b - \varsigma}{\eta_1}\right)$$

Data: Canadian national net financial positions

% of annual GDP	Households etc.	Non-financial firms	banks
currency and deposits	83.2	23.3	-106.5
debt securities	50.6	-27.7	-22.8
loans	-89.0	-35.4	124.5

1. integrate firms' "debt securities" with that of HHs', leaving none held by firms
2. integrate firms' "deposits and currencies" with that of HHs', leaving none held by firms
3. remove the loans to households (mortgage and consumer credit) from both HHs' and banks' balance sheets
4. adjust "currency and deposits" and "debt securities" so that their relative size does not change and that their total volume implies banks' leverage ratio of around 12

% of annual GDP	Households etc.	Non-financial firms	banks
currency and deposits	26.8	0	-26.8
debt securities	5.7	0	-5.7
loans	0	-35.4	35.4

These numbers look pretty small. We will pull the bank regulatory data to see the size of business loans relative to GDP.

Data: others

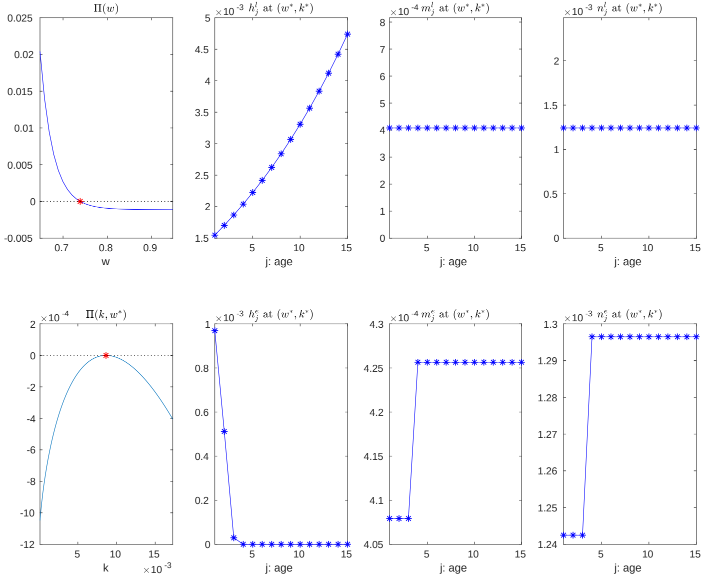
moment	value	time	source
working capital loans/term loans	0.797	2019Q4	RAPID 2
intermediate goods/GDP	1.11	2016	StatCan Table: 36-10-0217-01
I/GDP, C/GDP	0.22, 0.78[1]	2019Q4	StatCan Table: 36-10-0104-01
K/GDP[2]	1.16-2.5	2018	StatCan Table: 36-10-0098-01
labor share	0.65	2017	Penn World Table/FRED
π^f/GDP[3]	0.078	2019Q4	StatCan Table: 36-10-0125-01
π^b/GDP[4]	0.006-0.01	2019Q4	OSFI P3 v1285

[1] As fractions of domestic final demand.
[2] Relative to annualized GDP: 2.5 with gross non-residential capital stock; 1.16 with net capital stock with geometric consumption of capital
[3] Non-financial corportation profits before tax.
[4] Net income attributable to interest-generating activities of Canadian banks.

Numerical example: parameters

	value	annualized
β^h	0.995	0.980
ν^d	0.005	-
σ^c	2.000	-
σ^d	2.000	-
α	0.141	-
γ	0.329	-
θ	0.520	-
$\alpha+\gamma+\theta$	0.990	-
$\alpha/(\alpha+\gamma)$	0.300	-
δ_1	0.005	0.020
δ_2	0.013	0.050
β^b	0.985	0.940
λ	0.054	0.200
γ^d	0.003	0.011
η_0	0.010	-
η_1	0.991	-
θ^b	0.080	-
ω	1.000	-

Numerical example: profit function and firm decisions



Numerical example: SS statistics

	value	annualized
q	0.995	0.980
q^d	0.998	0.991
q^w	0.979	0.919
r^ℓ	0.021	0.089
$(\kappa^f + k^*)x_0^\ell$/GDP	0.017	-
#loan firms/#equity firms	0.093	-
work. cap. loan ratio (loan/equity)	12.875	-
employment ratio (loan/equity)	0.091	-
interm. goods ratio (loan/equity)	0.091	-
κ^f/k^*	0.144	-
ℓ^n/ℓ	0.063	-
tax on banks/GDP	0.017	-

	model	data[5]
K^f/GDP	2.943	2.500
I^f/GDP	0.215	0.220
C^h/GDP	0.785	0.780
$w(\widehat{N}^f + n_0^\ell x_0^\ell)$/GDP	0.681	0.650
$(\widehat{M}^f + m_0^\ell x_0^\ell)$/GDP	1.076	1.110
π^f/GDP	0.084	0.060
π^b/GDP	0.002	0.006-0.01
$(C^h - wN^h)$/GDP	0.108	0.130
B/GDP	0.109	0.057
D/GDP	0.265	0.268
work. cap. loans/term loans	0.422	0.797
total loans/GDP	0.408	0.354
leverage ratio	12.005	12.000

Some observations:

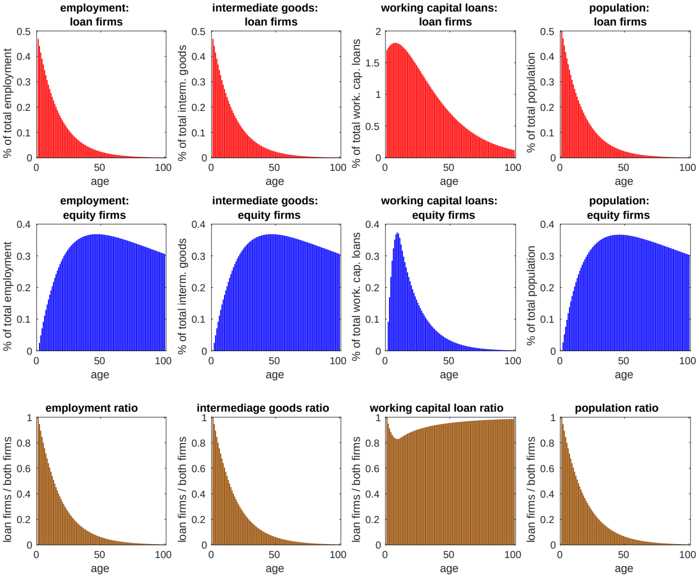
a. the fraction of loan firms is low.

b. working capital loans/term loans is below target.

c. the lending rate may be a bit too high [we'll collect data on this]

[5] Annualized where applicable.

Numerical example: beginning-of-period distribution by age



Parameters: a smaller λ?

	value	annualized
β^h	0.995	0.980
ν^d	0.021	-
σ^c	2.000	-
σ^d	2.000	-
α	0.141	-
γ	0.329	-
θ	0.520	-
$\alpha+\gamma+\theta$	0.990	-
$\alpha/(\alpha+\gamma)$	0.300	-
δ_1	0.005	0.020
δ_2	0.013	0.050
β^b	0.985	0.940
λ	0.031	0.120
γ^d	0.003	0.011
η_0	0.010	-
η_1	0.991	-
θ^b	0.080	-
ω	1.000	-

The change in v^d is to make the D/B ratio comparable to the data.

SS statistics: smaller λ

	value	annualized
q	0.995	0.980
q^d	0.998	0.991
q^w	0.979	0.919
r^ℓ	0.021	0.089
$(\kappa^f + k^*)x_0^\ell$ /GDP	0.016	-
#loan firms/#equity firms	0.160	-
work. cap. loan ratio (loan/equity)	38.119	-
employment ratio (loan/equity)	0.154	-
interm. goods ratio (loan/equity)	0.154	-
κ^f / k^*	0.135	-
ℓ^n / ℓ	0.038	-
tax on banks/GDP	0.030	-

	model	data[6]
K^f /GDP	2.756	2.500
I^f /GDP	0.201	0.220
C^h /GDP	0.799	0.780
$w(\widehat{N}^f + n_0^\ell x_0^\ell)$/GDP	0.682	0.650
$(\widehat{M}^f + m_0^\ell x_0^\ell)$/GDP	1.078	1.110
π^f /GDP	0.079	0.060
π^b /GDP	0.004	0.006-0.01
$(C^h - wN^h)$/GDP	0.121	0.130
B/GDP	0.118	0.057
D/GDP	0.556	0.268
work. cap. Loans/Term Loans	0.700	0.797
total loans/GDP	0.736	0.354
leverage ratio	12.005	12.000

a. the fraction of loan firms increases as maturity gets longer with lower λ

b. working capital loans/term loans closer to data

c. however, total loans/GDP increases; B/GDP↑ needed to finance this (D is tied to C through preference)

d. roughly, term loans and working capital loans increased with the change in λ as follows

$$\text{total loans/GDP} \simeq \underbrace{(K/GDP) * (\#\text{loan firms}/\#\text{equity firms})}_{\text{term loans/GDP}} *(1 + \text{working capital loans/term loans})$$

$$= 2.74 * \underbrace{0.16}_{\substack{\vee \\ 0.093}} * \underbrace{(1 + 0.7)}_{\substack{\vee \\ 1.422}} = 0.747 \simeq 0.736 > 0.408$$

[6] Annualized where applicable.

Still remaining tasks:

- Adding $c^v > 0$
- Transition dynamics: wage rigidity implies $w_t - w_{ss} = \eta^w (n_t - n_{ss})$
    - Step 1: equity firms only
    - Step 2: add loan firms

Transition dynamics: equity firms only

Households work as demanded by firms under wage rigidity

$$V_t(a_{t-1}, N^f_{t-1}) = \max_{c_t, b_t, d_t} \quad u(c_t) + \nu(d_t) + \beta^h V_{t+1}\left(a_t, (1 - \delta_{1,t})(1 - \delta_3) N^f_t\right)$$

subject to

$$c_t + q_t \; b_t + q^d_t \; d_t = a_{t-1} + w_{t-1} \; N^f_{t-1} + \pi^f_t + \pi^b_t + T_t$$
$$a_t = b_t + d_t$$

FOCs:

$$q_t = \beta^h \frac{u_{c,t+1}}{u_{c,t}}$$
$$q^d_t = \frac{\nu_{d,t}}{u_{c,t}} + q_t$$

Banks that provide only working capital loans

- For now, we omit the loan issuance costs but there is a cost when dividends deviate from SS.

$$V_t^b\left(a_t^b\right) = \max_{c_t^b, h_t^b, B_t^b, D_t^b} c_t^b + \beta^b \; V_{t+1}^b\left(a_{t+1}^b\right) \qquad \text{s.t.}$$

$$(TA) \qquad a_{t+1} = (1-\delta_{1,t})\; h_t^b - B_t^b - D_t^b$$

$$(BC) \qquad c_t^b + \kappa^B (c^b - c_{ss}^b)^2 + q_t^w \; h_t^b + \gamma^d D_t^b + \phi(\varsigma_t)\, q_t^w h_t^b \; + \iota_t \leq a_t + q_t \; B_t^b + q_t^d D_t^b$$

$$(KR) \qquad \varsigma_t = \frac{q_t^w h_t^b - q_t B_t^b - q_t^d D_t^b}{\omega \; q_t^w \; h_t^b}$$

Banks' FOCs:

Banks' FOCs imply

$$q_t^d = q_t + \frac{\gamma^d}{1+\phi_{\varsigma,t}/\omega} \quad \Rightarrow q_t^d \text{ makes banks indifferent between } D_t \text{ and } B_t \tag{30}$$

$$\frac{q_t}{\kappa^B(c_t^b - c_{ss}^b)}\left(1+\frac{\phi_{\varsigma,t}}{\omega}\right) = \frac{\beta^b}{\kappa^B(c_{t+\mathbf{1}}^b - c_{ss}^b)} \quad \Rightarrow \text{dynamic relationship when } \kappa^B > 0 \tag{31}$$

$$q_t^w = \frac{(1-\delta_{\mathbf{1},t})q_t\,(1+\phi_{\zeta,t}/\omega)}{1+\phi(\varsigma_t)+(1-\omega\varsigma_t)\frac{\phi_\varsigma}{\omega}} \quad \Rightarrow \text{ the default rate and the regulation determine } q_t^w \tag{32}$$

With $\kappa^B > 0$, the bank's problem is no longer a static one as in the SS.

A simplified version of the firms' problem

- We'll start from a special version of the general problem for tractability. A special version has firms that always borrow working capital loans from banks. They pay out dividends while borrowing.
- The production function exhibits DRS and has the Cobb-Douglas form. This allows us to obtain a closed-form solution to firms' decisions that cleanly separates the contribution of capital and other aggregate effects. This greatly simplifies aggregation.
-

Notations for firms' problem

- t: time
- j: age
- state variables of firms of age j in period t:
  1. $k_{0,t-j}$: capital chosen j periods ago as a new firm
  2. $(m_{j-1,t-1}, n_{j-1,t-1}, h_{j-1,t-1})$: determined in period $t-1$ at age $j-1$.
     - $(1-\delta_3)n_{j-1,t-1}$ is the effective employment used for production
     - firms save $c^v(1-\delta_3)n_{j-1,t-1}$ in vacancy posting costs
- choice variables of firms of age j in period t: $(m_{j,t}, n_{j,t}, h_{j,t})$
- $x_{j,t}$: the beginning-of-period measure of firms of age j in period t

$$\Omega_t(k_{\mathbf{0},t-j}, m_{j-\mathbf{1},t-\mathbf{1}}, n_{j-\mathbf{1},t-\mathbf{1}}, h_{j-\mathbf{1},t-\mathbf{1}}) =$$

$$\max_{n_{j,t}, m_{j,t}, h_{j,t}} \underbrace{z_t k_{\mathbf{0},t-j}^{\alpha} \left[(1-\delta_{\mathbf{3}}) n_{j-\mathbf{1},t-\mathbf{1}}\right]^{\gamma} m_{j-\mathbf{1},t-\mathbf{1}}^{\theta}}_{\textbf{Cobb-Douglas}} - h_{j-\mathbf{1},t-\mathbf{1}} - \delta_{\mathbf{2}} k_{\mathbf{0},t-j} + c^{v}(1-\delta_{\mathbf{3}}) n_{j-\mathbf{1},t-\mathbf{1}}$$

$$-\left(w_t + c^{v}\right) n_{j,t} - m_{j,t} + h_{t,j}\ q_t^{w} + q_t(1-\delta_{\mathbf{1},t})\ \Omega_{t+\mathbf{1}}(k_{\mathbf{0},t-j}, m_{j,t}, n_{j,t}, h_{j,t})$$

subject to

$$m_{j,t} + (w_t + c^{v}) n_{j,t} = q_t^{w} h_{j,t}$$

$$div_{j,t} = z_t k_{t-j}^{\alpha} \left[(1-\delta_{\mathbf{3}}) n_{j-\mathbf{1},t-\mathbf{1}}\right]^{\gamma} m_{j-\mathbf{1},t-\mathbf{1}}^{\theta} - h_{j-\mathbf{1},t-\mathbf{1}} - \delta_{\mathbf{2}} k_{\mathbf{0},t-j} + c^{v}(1-\delta_{\mathbf{3}}) n_{j-\mathbf{1},t-\mathbf{1}} \geq 0$$

Closed-form solutions:

$$n_{j,t}^{w} = \left[q_t^{w} (1 - \delta_{\mathbf{3}})^{\gamma} z_{t+\mathbf{1}} \left(\frac{\gamma}{w_t + c^{v}\left(1 - q_t^{w}(1-\delta_{\mathbf{3}})\right)} \right)^{\mathbf{1}-\theta} \theta^{\theta} \right]^{\frac{\mathbf{1}}{\mathbf{1}-\gamma-\theta}} k_{\mathbf{0},t-j}^{\frac{\alpha}{\mathbf{1}-\gamma-\theta}} \equiv A_t^{n} k_{\mathbf{0},t-j}^{\frac{\alpha}{\mathbf{1}-\gamma-\theta}}$$

$$m_{j,t}^{w} = \frac{\theta n_{j,t}^{w}}{\gamma} \left\{ w_t + c^{v} \left[1 - q_t^{w}(1-\delta_{\mathbf{3}})\right] \right\} \equiv A_t^{m} k_{\mathbf{0},t-j}^{\frac{\alpha}{\mathbf{1}-\gamma-\theta}}$$

$$h_{j,t} = \frac{m_{j,t}^{w} + (w_t + c^{v}) n_{j,t}^{w}}{q_t^{w}} \equiv A_t^{h} k_{\mathbf{0},t-j}^{\frac{\alpha}{\mathbf{1}-\gamma-\theta}}$$

$$y_{j,t} = z_t n_{j-\mathbf{1}}^{\gamma} m_{j-\mathbf{1},t-\mathbf{1}}^{\theta} k_{\mathbf{0},t-j}^{\alpha} = z_t \left[(1-\delta_{\mathbf{3}})\ A_{t-\mathbf{1}}^{n}\right]^{\gamma} \left(A_{t-\mathbf{1}}^{m}\right)^{\theta} k_{\mathbf{0},t-j}^{\frac{\alpha}{\mathbf{1}-\gamma-\theta}} \equiv A_t^{y} k_{\mathbf{0},t-j}^{\frac{\alpha}{\mathbf{1}-\gamma-\theta}}$$

New firms' entry decisions

- The mutual fund pays $k_{\mathbf{0},t} + \kappa^f$ to gain the ownership of a firm.

- A new firm borrows working capital loans from a bank, which depends on the firm size.

$$k^*_{\mathbf{0},t} = \arg\max_k \left\{ -k + q_t(1-\delta_{\mathbf{1},t})\Omega_{t+\mathbf{1}} \left(k,\ m^w_{\mathbf{0},t}(k),\ n^w_{\mathbf{0},t}(k),\ \frac{m^w_{\mathbf{0},t}(k) + (w_t + c^v) n^w_{\mathbf{0},t}(k)}{q^w_t} \right) \right\}$$

Given the forms of decision rules, we can write the value function as follows:

$$\Omega_{j,t} = v^p_t \; k_{0,t\text{-}j}^{\frac{\alpha}{1-\gamma-\theta}} + v^l_t \; k_{0,t\text{-}j}$$

Substituting this into the firms' dynamic program, we obtain the following:

$$\begin{aligned} v^p_t &= A^y_t - A^h_{t\text{-}1} + c^v(1-\delta_3)A^n_{t\text{-}1} + q_t(1-\delta_1)v^p_{t+1} \\ v^l_t &= -\delta_2 + q_t(1-\delta_1)v^l_{t+1} \end{aligned}$$

Using this result, the optimal $k_{0,t}$ satisfies the following FOC:

$$1 = q_t(1-\delta_1)\left[\frac{\alpha}{1-\gamma-\theta} v^p_{t+1} k_{0,t}^{\frac{\alpha+\gamma+\theta-1}{1-\gamma-\theta}} + v^l_{t+1}\right]$$

The zero-profit condition is expressed as

$$\kappa^f + k_{0,t} = q_t(1-\delta_1)\left[v^p_{t+1} k_{0,t}^{\frac{\alpha}{1-\gamma-\theta}} + v^l_{t+1} k_{0,t}\right]$$

Aggregation (special version) with Cobb-Douglas prod. fn.

Aggregate capital

$$K_t^f = \sum_{j=1}^{\infty} x_{j,t} k_{0,t-j} = \sum_{j=1}^{\infty} (1-\delta_1) x_{j-1,t-1} k_{0,t-j} = (1-\delta_1)\left[\sum_{j=1}^{\infty} x_{j,t-1} k_{0,t-1-j} + x_{0,t-1} k_{0,t-1}\right]$$
$$= (1-\delta_1)\left[K_{t-1}^f + x_{0,t-1} k_{0,t-1}\right]$$

Useful expression Let $F_t^{xk} \equiv \sum_{j=1}^{\infty} x_{j,t} k_{0,t-j}^{\frac{\alpha}{1-\gamma-\theta}}$.

$$F_t^{xk} = (1-\delta_1)\left[F_{t-1}^{xk} + x_{0,t-1} k_{0,t-1}^{\frac{\alpha}{1-\gamma-\theta}}\right]$$

Other aggregates

$$\widehat{N}_t = \sum_{j=1}^{\infty} x_{j,t} n_{j,t} = \sum_{j=1}^{\infty} x_{j,t} A_t^n k_{0,t-j}^{\frac{\alpha}{1-\gamma-\theta}} = A_t^n F_t^{xk}$$

$$\widehat{H}_t = \sum_{j=1}^{\infty} x_{j,t} h_{j,t} = \sum_{j=1}^{\infty} x_{j,t} A_t^h k_{0,t-j}^{\frac{\alpha}{1-\gamma-\theta}} = A_t^h F_t^{xk}$$

$$\Upsilon_t = \sum_{j=1}^{\infty} x_{j,t} y_{j,t} = \sum_{j=1}^{\infty} x_{j,t} A_t^y k_{0,t-j}^{\frac{\alpha}{1-\gamma-\theta}} = A_t^y F_t^{xk}$$

Other equilibrium conditions

[Goods market clearing] $$Y_t = C_t + I_t$$

[Sticky wage dynamics] $$w_t - w_{ss} = \eta^w (N_t^f - N_{ss})$$

[Zero profit condition] $$\kappa^f = -k_{\mathbf{0},t} + q_t(1 - \delta_{\mathbf{1},t})\Omega_{t+\mathbf{1}}(k_{\mathbf{0},t}, m_{\mathbf{0},t}, n_{\mathbf{0},t}, h_{\mathbf{0},t})$$

The solution method

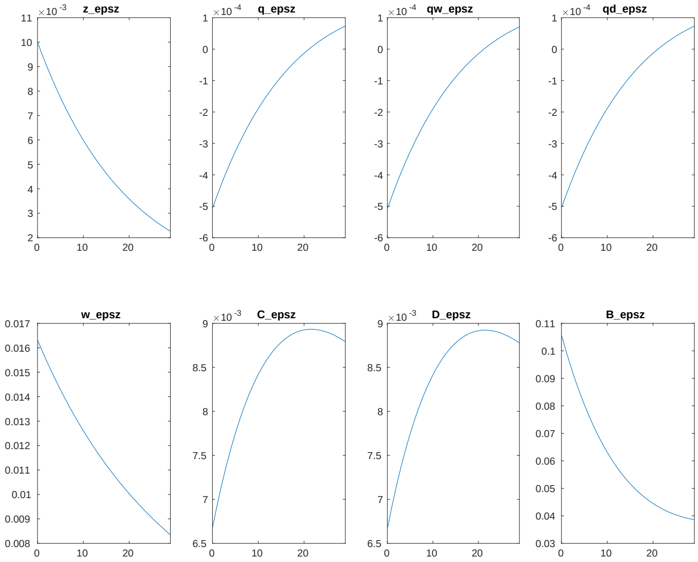
- We use dynare and let it linearize the non-linear system.

- To begin with, we examine IRFs following a positive TFP shock.

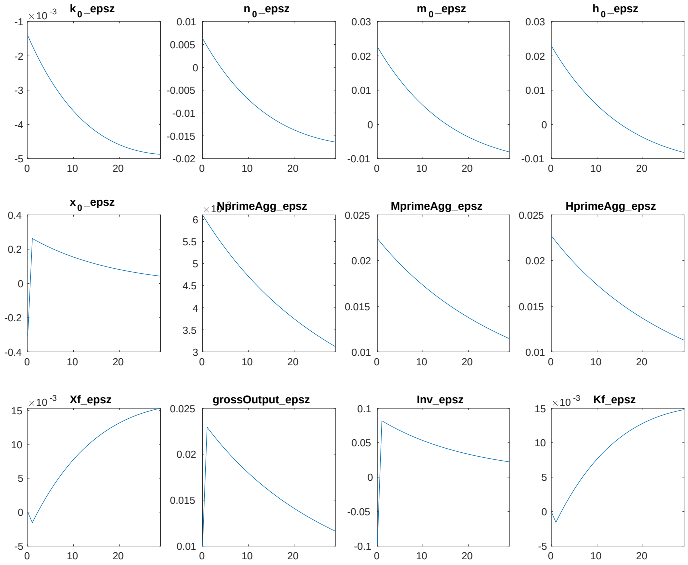
$$z_t - z_{ss} = \rho_z \left(z_{t-1} - z_{ss} \right) + \epsilon_t^z$$

- Each graph is expressed in % deviation from the SS except the capital ratio and the penalty rate of banks. 0.01 on the y-axis should be interpreted as 1% deviation from the SS.

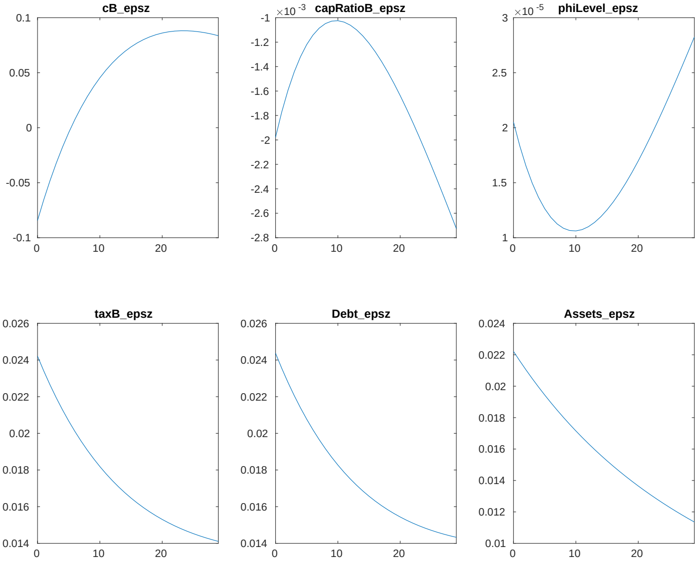
TFP shock $\rho_z = 0.95, \kappa^B = 30, \eta^w = 1$: prices & households



TFP SHOCK $\rho_z = 0.95, \kappa^B = 30, \eta^w = 1$: FIRMS



TFP SHOCK $\rho_z = 0.95, \kappa^B = 30, \eta^w = 1$: BANKS



Problems (Jan 2021)

- Firm entry and investment decreases on impact of the shock. Due to lags in the expansion of production input, the resources (gross output) in the initial period is limited at the aggregate level despite the increase in TFP. To meet the increased demand for labor and intermediate goods at the indivisual level, the equilibrium entry declines temporarily.

- The individual capital demand at entry (k_0) declines not only on impact but also over time.

- In order to address this issue, we add the following new features:

1. CES production function to allow for complementarity between intermediate goods and other factors:

$$y = z\left[(1-\xi)\left(k^{\alpha_1}\left[(1-\delta_3)n\right]^{\alpha_2}\right)^{-\nu} + \xi m^{-\alpha_3 \nu}\right]^{-\frac{1}{\nu}},$$

where $\alpha_1(1-\xi) + \alpha_2(1-\xi) + \alpha_3\xi < 1$.[7]

2. Additional cost of intermediate goods: $\frac{\kappa^M}{2}\left(\frac{m_{j,t} - m_{ss}}{m_{ss}}\right)^2 m_{ss}$

[7] $\lim_{\nu\to 0} y = zk^{\alpha_1(1-\xi)}[(1-\delta_3)n]^{\alpha_2(1-\xi)}m^{\alpha_3\xi}$, $\lim_{\nu\to\infty} = z\min\left\{k^{\alpha_1}[(1-\delta_3)]^{\alpha_2}, m^{\alpha_3}\right\}$.

Approach to solve the new model

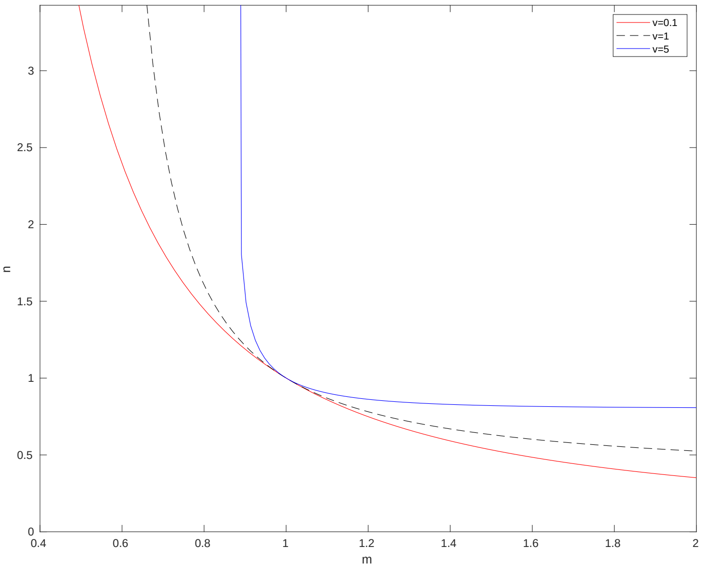
- With either of these additional features, even though decision rules are still static, we can no longer obtain closed-form expressions for individual decisions. Hence, we can no longer express firm value and aggregate firm decisions compactly in the non-linear model.

- In the linearized system, however, we can still separate firm decisions into the effect of age-specific capital and that of aggregate prices and shocks. This allows us to derive compact expressions for firms' aggregate decisions and their value functions.

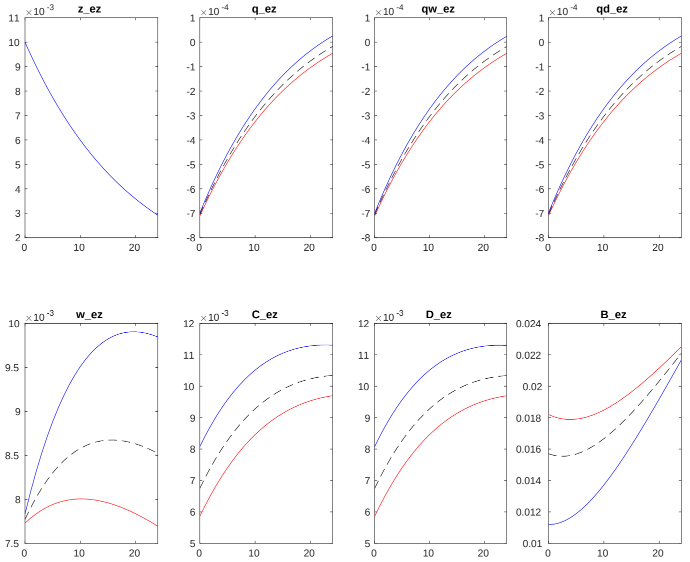
- The derivation of linearized aggregate decisions and linearized value function requires some tedious semi-manual work outside dynare (details explained in a separate note).

κ^B	κ^M	η^W	ν	ξ	$\alpha_{\mathbf{1}}$	$\alpha_{\mathbf{2}}$	$\alpha_{\mathbf{3}}$
30.0	2.0	1.0	0.1, 1.0, 5.0	0.6	0.25,	0.75	0.93

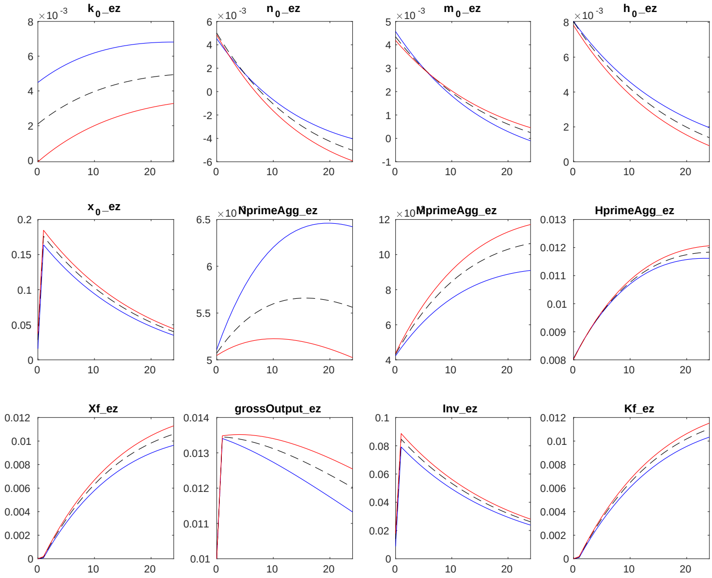
Production function: $y = k = z = 1$



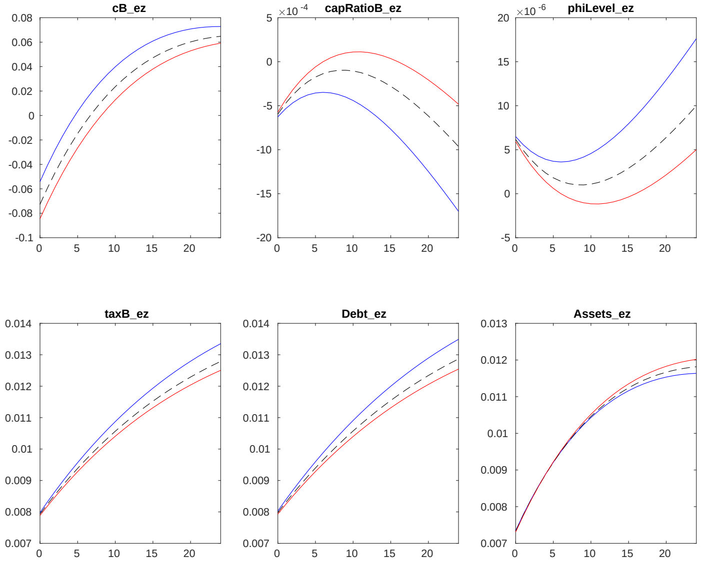
TFP SHOCK $(\nu = 0.1, 1, 5)$: PRICES & HOUSEHOLDS



TFP shock ($\nu = 0.1, 1, 5$): firms



TFP shock ($\nu = 0.1, 1, 5$): banks



TFP shock: observations

- The cost of intermediate goods production dampens the pace of increase in employment and intermediate goods per firm, which allows a positive measure of firms to enter the market on impact of the shock even though production input has not expanded yet at that point to take advantage of higher production efficiency.
- The complementarity between intermediate goods and other production input *increases* the individual demand for capital at entry (k_0) and over time for sufficiently high values of ν.
- Similarly, the increase in the demand for employment is dampened and more persistent due to the effects of intermediate goods production cost and the complementarity.

Funding shock on banks

- We characterize funding shocks as exogenous disturbances that cause troubles for bank's funding and repayment.
- First, a funding shock causes a fraction ι_t of bank's liability $(qB + q^d D)$ to be consumed before being used for banking activities.
- Second, households lose a fraction $\xi^b \iota_t$ of repayment by the bank at the beginning of the next period. However, deposit insurance funded by lump-sum taxes on households ensures that deposits will be fully repaid. This shock tilts HH savings towards deposits and away from the wholesale funding.

Banks with funding shocks

$$V_t^b\left(a_t^b\right) = \max_{c_t^b, h_t^b, B_t^b, D_t^b} c_t^b + \beta^b \; V_{t+1}^b\left(a_{t+1}^b\right) \qquad \text{s.t.}$$

(TA) $$a_{t+1} = (1 - \delta_{1,t}) \; h_t^b - B_t^b - D_t^b$$

(BC) $$c_t^b + \frac{\kappa^B}{2}(c^b - c_{ss}^b)^2 + q_t^w \; h_t^b + \gamma^d D_t^b + \phi(\varsigma_t) \, q_t^w h_t^b \; \leq a_t + (1 - \iota_t)\left(q_t \; B_t^b + q_t^d D_t^b\right)$$

(KR) $$\varsigma_t = \frac{q_t^w h_t^b - q_t B_t^b - q_t^d D_t^b}{\omega \; q_t^w \; h_t^b}$$

$(SHOCK)$ $$\iota_t = \rho_\iota \iota_{t-1} + \varepsilon_\iota, \quad \iota_{ss} = 0$$

Households with funding shocks

$$V_t(a_{t-1}, N^f_{t-1}) = \max_{c_t, b_t, d_t} \quad u(c_t) + \nu(d_t) + \beta^h V_{t+1}\left(a_t, (1 - \delta_{1,t})(1 - \delta_3)N^f_t\right)$$

subject to

$$c_t + q_t \; b_t + q^d_t \; d_t = a_{t-1} + w_{t-1} \; N^f_{t-1} + \pi^f_t + \pi^b_t + T_t$$
$$a_t = (1 - \xi^b \iota_t) b_t + d_t$$

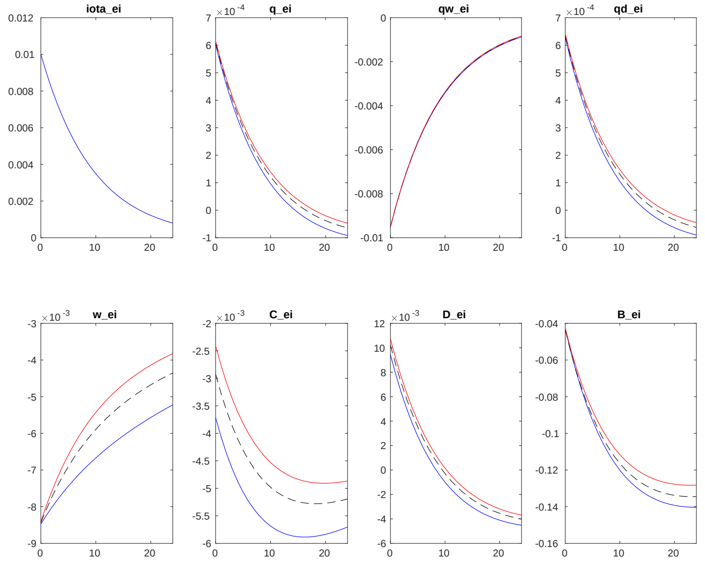
FOCs:

$$q_t = \beta^h \frac{u_{c,t+1}}{u_{c,t}} (1 - \xi^b \iota)$$
$$q^d_t = \frac{\nu_{d,t}}{u_{c,t}} + \frac{q_t}{1 - \xi^b \iota}$$

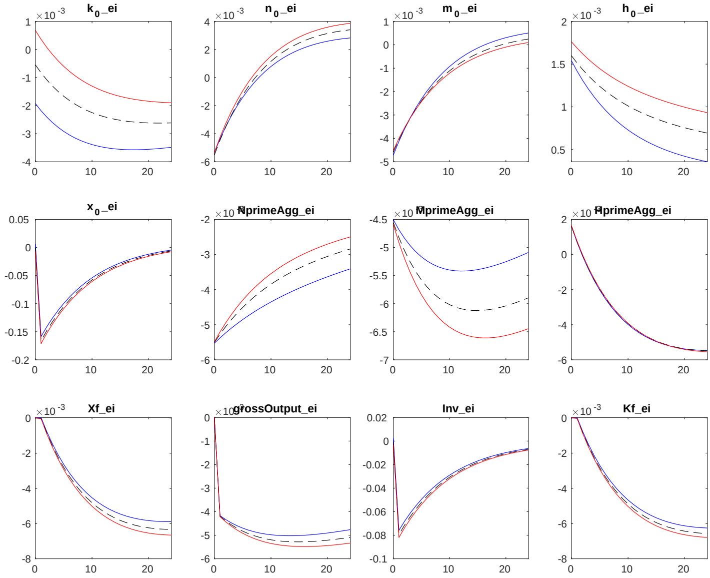
New parameters:

ξ^b	ρ^ι
0.01	0.9

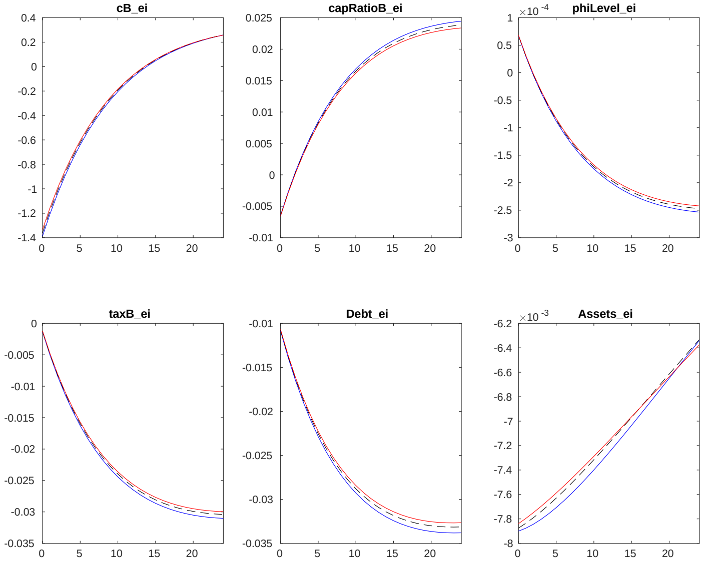
FUNDING SHOCK ($\nu = 0.1, 1, 5$): PRICES & HOUSEHOLDS



Funding shock ($\nu = 0.1, 1, 5$): firms



Funding shock ($\nu = 0.1, 1, 5$): banks



Funding shock: observations

- A funding shock leads to a drop in quantities as financial intermediation becomes costly.
- The spreaad beween the lending rate ($1/q^w$) and the funding rates ($1/q$ or $1/q^d$) widens.
- The bank dividend response is large and sensitive to the choice of κ^b.
- Also, responses of D and B are sensitive to the choice of ξ^b.