Course in Heterogeneity: Econ 081

Notes on Measure Theory, Industry Equilibrium and the Aiyagari Model

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Contents

1 Measure Theory

This section will be a quick review of measure theory to be able to use it in the subsequent sections. In macroeconomics we encounter the problem of aggregation often and it's crucial that we do it in a reasonable way. Measure theory is a tool that tells us when and how we could do so. Let us start with some definitions on sets.

Definition 1 *For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).*

Remark 1 *Note that, in this section we will follow the convention of notations as following*

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition 2 *A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if*

1. *$S, \emptyset \in \mathcal{S}$;*
2. *$A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements); and,*
3. *for $\{B_i\}_{i\in\mathbb{N}}$, $B_i \in \mathcal{S}$ for all i implies $\bigcap_{i\in\mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections).*

Example 1

1. *The power set of S (i.e. all the possible subsets of a set S), is a σ-algebra in S.*
2. *$\{\emptyset, S\}$ is a σ-algebra in S.*

3. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$, *where* $S_{1/2}$ *means the lower half of* S *(imagine* S *as an closed interval in* $\mathbb{R}$*), is a* σ*-algebra in* S.

4. *If* $S = [0, 1]$*, then*

$$\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$$

is not *a* σ*-algebra in* S. *But*

$$\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$$

is a σ*-algebra in* S.

Why do we need this σ-algebra? The answer is it defines which sets may be considered as "events": things that could happen. Elements not in it may have no properly defined measure. Basically, σ-algebra is the "patch" that lets us avoid some pathological behaviors of mathematics, namely non-measurable sets. We are now ready to define a measure.

Definition 3 *Suppose* $\mathcal{S}$ *is a* σ*-algebra in* S. *A measure is a function* $x : \mathcal{S} \to \mathbb{R}_+$*, that satisfies*

1. $x(\emptyset) = 0$;

2. $B_1, B_2 \in \mathcal{S}$ *and* $B_1 \cap B_2 = \emptyset$ *implies* $x(B_1 \cup B_2) = x(B_1) + x(B_2)$ *(additivity); and,*

3. $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ *and* $B_i \cap B_j = \emptyset$*, for all* $i \neq j$*, implies* $x(\cup_i B_i) = \sum_i x(B_i)$ *(countable additivity).*[1]

Put simply, a measure is just a way to assign each possible "event" a non-negative real number. A set S, a σ-algebra in it, $\mathcal{S}$, and a measure on $\mathcal{S}$, define a measure space, $(S, \mathcal{S}, x)$.

Definition 4 *Borel* σ*-algebra is a* σ*-algebra generated by the family of all open sets (generated by a topology).*

[1] Countable additivity means that the measure of the union of countable disjoint sets is the sum of the measure of these sets.

Since a Borel σ-algebra contains all the subsets generated by the intervals, you can recognize any subset of a set using Borel σ-algebra. In other words, Borel σ-algebra corresponds to complete information.

Definition 5 *A probability (measure) is a measure with the property that* $x(S) = 1$*.*

Definition 6 *Given a measure space* $(S, \mathcal{S}, x)$*, a function* $f : S \to \mathbb{R}$ *is measurable (with respect to the measure space) if, for all* $a \in \mathbb{R}$*, we have*

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

One way to interpret a σ-algebra is that it describes the information available based on observations; a structure to organize information, and how fine are the information that we receive. Suppose that S is comprised of possible outcomes of a dice throw. If you have no information regarding the outcome of the dice, the only possible sets in your σ-algebra can be $\emptyset$ and S. If you know that the number is even, then the smallest σ-algebra given that information is $\mathcal{S} = \{\emptyset, \{2, 4, 6\}, \{1, 3, 5\}, S\}$. Measurability has a similar interpretation. A function is measurable with respect to a σ-algebra $\mathcal{S}$, if it can be evaluated under the current measure space $(S, \mathcal{S}, x)$.

Example 2 *Suppose* $S = \{1, 2, 3, 4, 5, 6\}$*. Consider a function* f *which maps the element 6 to a number 1 (i.e.* $f(6) = 1$*) and any other elements to -100. Then* f *is NOT measurable with respect to* $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$*. Why? Consider* $a = 0$*, then* $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$*. But this set is not in* $\mathcal{S}$*.*

We can also generalize Markov transition matrix to any measurable space. This is what we do next.

Definition 7 *A function* $Q : \mathcal{S} \times S \to [0, 1]$ *is a transition probability if*

1. $Q(\cdot, s)$ *is a probability measure for all* $s \in S$*; and,*

2. $Q(B, \cdot)$ *is a measurable function for all* $B \in \mathcal{S}$*.*

Intuitively, given $B \in \mathcal{S}$ and $s \in S$, $Q(B, s)$ gives the probability of being in set B tomorrow, given that the state is s today. Consider the following example: a *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on the set $S = \{1, 2, 3\}$, with the σ-algebra $\mathcal{S} = P(S)$ (where $P(S)$ is the power set of S). If Γ_{ij} denotes the probability of state j happening, given a present state i, then

$$Q(\{1, 2\}, 3) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 .$$

As another example, suppose we are given a measure x on $\mathcal{S}$; x_i gives us the fraction of type i, for $i \in S$. Given the previous transition function, we can calculate the fraction of types tomorrow using the following formulas:

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words

$$\mathbf{x}' = \Gamma^T \mathbf{x},$$

where $\mathbf{x}^T = (x_1, x_2, x_3)$.

To extend this idea to a general case with a general transition function, we define an *updating operator* as $T(x, Q)$, which is a measure on S with respect to the σ-algebra $\mathcal{S}$, such that

$$\begin{aligned} x'(B) =& T(x, Q)(B) \\ =& \int_S Q(B, s)\, x(ds), \quad \forall B \in \mathcal{S}. \end{aligned}$$

A stationary distribution is a fixed point of T, that is x^* so that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

We know that, if Q has nice properties,[2] then a unique stationary distribution exists (for example, we discard *flipping* from one state to another), and

$$x^* = \lim_{n \to \infty} T^n(x_0, Q),$$

for any x_0 in the space of measures on $\mathcal{S}$.

Exercise 1 *Consider unemployment in a very simple economy (we have an exogenous transition matrix). There are two states; employed and unemployed. The transition matrix is given by*

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

[2] See Chapter 11 in **?**.

2 Industry Equilibrium

2.1 Preliminaries

Now we are going to study a type of models initiated by **?**. We will abandon the general equilibrium framework from the previous section to study the dynamics of distribution of firms in a partial equilibrium environment.

To motivate things, let's start with the problem of a single firm that produces a good using labor input according to a technology described by the production function f. Let us assume that this function is increasing, strictly concave, with $f(0) = 0$. A firm that hires n units of labor is able to produce $sf(n)$, where s is a productivity parameter. Markets are competitive, in the sense that a firm takes prices as given and chooses n in order to solve

$$\pi(s, p) = \max_{n \geq 0} \{psf(n) - wn\}.$$

The first order condition implies that in the optimum, n^*,

$$psf_n(n^*) = w.$$

Let us denote the solution to this problem as a function $n^*(s, p)$.[3] Given the above assumptions, n^* is an increasing function of s (i.e. more productive firms have more workers), as well as p.

Suppose now there is a mass of firms in the industry, each associated with a productivity parameter $s \in S \subset \mathbb{R}_+$, where $S := [\underline{s}, \bar{s}]$. Let $\mathcal{S}$ denote a σ-algebra on S (Borel σ-algebra for instance). Let x be a measure defined over the space $(S, \mathcal{S})$ that describes the cross sectional distribution of productivity among firms. Then, for any $B \subset S$ with $B \in \mathcal{S}$, $x(B)$ is the mass of firms having productivities in S.

We will use x to define statistics of the industry. For example, at this point, it is convenient to define

[3] As we declared in advance, this is a partial equilibrium analysis. Hence, we ignore the dependence of the solution on w to focus on the determination of p.

the aggregate supply of the industry. Since individual supply is just $sf\left(n^*\left(s,p\right)\right)$, the aggregate supply can be written as[4]

$$Y^S\left(p\right) = \int_S sf\left(n^*\left(s,p\right)\right)x\left(ds\right).$$

Observe that Y^S is a function of the price p; for any price, p, $Y^S\left(p\right)$ gives us the supply in this economy.

Exercise 2 *Search Wikipedia for an index of concentration in an industry, and adopt it for our economy.*

Suppose now that the demand of the market is described by some function $Y^D\left(p\right)$. Then the equilibrium price, p^*, is determined by the market clearing condition

$$Y^D\left(p^*\right) = Y^S\left(p^*\right). \tag{1}$$

So far, everything is too simple to be interesting. The ultimate goal here is to understand how the object x is determined by the fundamentals of the industry. Hence, we will be adding tweaks to this basic environment in order to obtain a theory of firms' distribution in a competitive environment. Let's start by allowing firms to die.

2.2 A Simple Dynamic Environment

Consider now a dynamic environment, in which the situation above repeats every period. Firms discount profits at rate r_t, which is exogenously given. In addition, assume that a single firm, in each period, faces a probability $1-\delta$ of disappearing! We will focus on *stationary equilibria*; i.e. equilibria in which the price of the final output p, the rate of return, r, and the productivity of firm, s, stay constant through time.

Notice first that firm's decision problem is still a static problem; we can easily write the value of an

[4] S in Y^S stands for supply.

incumbent firm as

$$\begin{aligned} V(s,p) &= \sum_{t=0}^{\infty}\left(\frac{\delta}{1+r}\right)^{t}\pi(s,p) \\ &= \left(\frac{1+r}{1+r-\delta}\right)\pi(s,p) \end{aligned}$$

Note that we are considering that p is fixed (therefore we can omit it from the expressions above). Observe that every period there is positive mass of firms that die. Therefore, how can this economy be in a stationary equilibrium? To achieve that, we have to assume that there is a constant flow of firms entering the economy in each period, as well.

As before, let x be the measure describing the distribution of firms within the industry. The mass of firms that die is given by $(1-\delta)x(S)$. We will allow these firms to be replaced by new entrants. These entrants draw a productivity parameter s from a probability measure γ.

One might ask what keeps these firms out of the market in the first place? If

$$\pi(s,p) = psf(n^*(s,p)) - wn^*(s,p) > 0,$$

which is the case for the firms operating in the market, then all the (potential) firms with productivities in S would want to enter the market!

We can fix this flaw by assuming that there is a fixed entry cost that each firm must pay in order to operate in the market, denoted by c^E. Moreover, we will assume that the entrant has to pay this cost before learning s. Hence the value of a new entrant is given by the following function:

$$V^E(p) = \int_S V(s,p)\gamma(ds) - c^E. \tag{2}$$

Entrants will continue to enter if V^E is greater than 0, and decide not to enter if this value is less than zero. As a result, stationarity occurs when V^E is exactly equal to zero (this is the *free entry* assumption, and we are assuming that there is an infinite number (mass) of prospective firms).

Let's analyze how this environment shapes the distribution of firms in the market. Let x_t be the cross sectional distribution of firms in period t. For any $B \subset S$, portion $1-\delta$ of the firms with productivity $s \in B$ will die, and that will attract some newcomers. Hence, next period's measure of firms on set B will be given by:

$$x_{t+1}(B) = \delta x_t(B) + m\gamma(B).$$

That is, mass m of firms would enter the market in $t+1$, and only fraction $\gamma(B)$ of them will have productivities in the set B. As you might suspect, this relationship must hold for every $B \in \mathcal{S}$. Moreover, since we are interested in stationary equilibria, the previous expression tells us that the cross sectional distribution of firms will be completely determined by γ.

If we let mapping T be defined by

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{3}$$

a stationary distribution of productivity is the fixed point of the mapping T; i.e. x^* with $Tx^* = x^*$, implying:

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}.$$

Now, note that the demand and supply relation in (1) takes the form:

$$y^d(p^*(m)) = \int_S sf(n^*(s,p))\, dx^*(s;m), \tag{4}$$

whose solution, $p^*(m)$, is continuous function under regularity conditions stated in **?**.

We have two equations, (2) and (4), and two unknowns, p and m. Thus, we can defined the equilibrium as:

Definition 8 *A stationary distribution for this environment consists of functions p^*, x^*, and m^*, that* *satisfy:*

1. $y^d(p^*(m)) = \int_S sf(n^*(s,p))\, dx^*(s;m)$;

2. $\int_s V(s,p)\gamma(ds) - c^E = 0$; *and,*

3. $x^*(B) = \delta x^*(B) + m^*\gamma(B), \quad \forall B \in \mathcal{S}$.

2.3 Introducing Exit Decisions

We want to introduce more (economic) content by making the exit of firms endogenous (a decision of the firm). One way to do so is to assume that the productivity of the firms follow a Markov process governed by a transition function, Γ. This would change the mapping T in Equation (3), as:

$$Tx(B) = \delta \int_S \Gamma(s,B)\, x(ds) + m\gamma(B), \quad \forall B \in \mathcal{S}.$$

But, this wouldn't add much economic content to our environment; firms still do not make any (interesting) decision. To change this, let's introduce cost of operation into the model; suppose firms have to pay c^v each period in order to stay in the market. In this case, when s is low, the firm's profit might not cover its cost of operation. So, the firm might decide to leave the market. However, firm has already paid (a sunk cost of) c^E, and, since s changes according to a Markov process, prospects of future profits might deter the firm from quitting. Therefore, negative profit in one period does not imply immediately that the firm's optimal choice is to leave the market.

By adding such a minor change, the solution will have a reservation productivity property under some conditions (to be discussed in the comment below). In words, there will be a minimum productivity, $s^* \in S$, above which it is profitable for the firm to stay in the market.

To see this, note that the value of a firm with productivity $s \in S$ in a period is given by

$$V(s,p) = \max \left\{ 0, \pi(s,p) + \frac{1}{(1+r)} \int_S \Gamma(s, ds') V(s', p) - c^v \right\}.$$

Exercise 3 *Show that the firm's decision takes the form of a reservation productivity strategy, in which, for some $s^* \in S$, $s < s^*$ implies that the firm would leave the market.*

In this case, the transition of the distribution of productivities on S will be:

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) x(ds), \quad \forall B \in \mathcal{S}.$$

A stationary distribution of the firms in this economy, x^*, is the fixed point of this equation.

Example 3 *How productive does a firm have to be, to be in the top 10% largest firms in this economy? The answer to this question is the solution to the following equation, $\hat{s}$:*

$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1 .$$

Then, the fraction of the labor force in the top 10% largest firms in this economy, is

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p) x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p) x^*(ds)}.$$

Exercise 4 *Compute the average growth rate of the smallest one third of the firms. What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?*

Comment 1 *To see that this will be the case you should prove that the profit before variable cost function $\pi(s,p)$ is increasing in s. Hence the productivity threshold is given by the s^* that satisfies the following condition:*

$$\pi(s^*, p) = c_v$$

for an equilibrium price p. Now instead of considering γ as the probability measure describing the distribution of productivities among entrants, you must consider $\widehat{\gamma}$ defined as follows

$$\widehat{\gamma}(B) = \frac{\gamma(B \cap [s^*, \overline{s}])}{\gamma([s^*, \overline{s}])}$$

for any $B \in \mathcal{S}$.

One might suspect that this is an ad hoc *way to introduce the exit decision. To make things more concrete and easier to compute, we will assume that s is a Markov process. To facilitate the exposition, let's make S finite and assume s has transition matrix Γ. Assume further that Γ is regular enough so that it has a stationary distribution γ^*. For the moment we will not put any additional structure on Γ.*

The operation cost c^v is such that the exit decision is meaningful. Let's analyze the problem from the perspective of the firm's manager. He has now two things to decide. First, he asks himself the question "Should I stay or should I go?". Second, conditional on staying, he has to decide how much labor to hire. Importantly, notice that this second decision is still a static decision. Later, we will introduce adjustment cost that will make this decision a dynamic one.

Let $\phi(s,p)$ be the value of the firm before having decided whether to stay or to go. Let $V(s,p)$ be the value of the firm that has already decided to stay. $V(s,p)$ satisfies

$$V(s,p) = \max_n \left\{ spf(n) - n - c^v + \frac{1}{1+r} \sum_{s' \in S} \Gamma_{ss'} \phi(s',p) \right\}$$

Each morning the firm chooses d in order to solve

$$\phi(s,p) = \max_{d \in \{0,1\}} dV(s,p)$$

Let $d^*(s,p)$ be the optimal decision to this problem. Then $d^*(s,p) = 1$ means that the firm stays in the market. One can alternatively write:

$$\phi(s,p) = \max_{d \in \{0,1\}} d \left[\pi(s,p) - c^v + \frac{1}{1+r} \sum_{s' \in S} \Gamma_{ss'} \phi(s',p) \right]$$

or even

$$\phi(s,p) = \max\left[\pi(s,p) - c^v + \frac{1}{1+r}\sum_{s'\in S}\Gamma_{ss'}\phi(s',p), 0\right] \tag{5}$$

All these are valid. Additionally, one can easily add minor changes to make the exit decision more interesting. For example, things like scrap value or liquidation costs will affect the second argument of the max *operator above, which so far is just zero.*

What about $d^*(s,p)$? Given a price, this decision rule can take only finitely many values. Moreover, if we could ensure that this decision is of the form "stay only if the productivity is high enough and go otherwise" then the rule can be summarized by a unique number $s^* \in S$. Without doubt, that would be very convenient, but we don't have enough structure to ensure that such is the case. Because, although the ordering of s (lower s are ordered before higher s) gives us that the value of s today is bigger than value of smaller s', depending on the Markov chain, on the other hand, the value of productivity level s tomorrow may be lower than the value of s' (note $s' < s$) tomorrow. Therefore we need some additional regularity conditions.

In order to get a cutoff rule for the exit decision, we need to add an assumption about the transition matrix Γ. Let the notation $\Gamma(s)$ indicate the probability distribution over next period state conditional on being on state s today. You can think of it as being just a row of the transition matrix. Take s and $\widehat{s}$. We will say that the matrix Γ displays first order stochastic dominance $(FOSD)$ if $s > \widehat{s}$ implies $\sum_{s'\leq b}\Gamma(s' \mid s) \leq \sum_{s'\leq b}\Gamma(s' \mid \widehat{s})$ for any $b \in S$. It turns out that $FOSD$ is a sufficient condition for having a cutoff rule. You can prove that by using the same kind of dynamic programming tricks that have been used in a different course for obtaining the reservation wage property in search problems. Try it as an exercise. Also note that this is just a sufficient condition.

Finally, we need to mention something about potential entrants. Since we will assume that they have to pay the cost c^E before learning their s, they can leave the industry even before producing anything. That requires us to be careful when we describe industry dynamics.

Now the law of motion becomes:

$$x'(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) \, x(ds), \quad \forall B \in \mathcal{S}.$$

2.4 Stationary Equilibrium

Now that we have all the ingredients in the table, let's define the equilibrium formally.

Definition 9 *A stationary equilibrium for this environment consists of a list of functions (ϕ, n^*, d^*), a price p^* and a measure x^* such that*

1. *Given p^*, the functions ϕ, n^*, d^* solve the problem of the incumbent firm*
2. $V^E(p^*) = 0$
3. *For any $B \in \mathcal{S}$ (assuming we have a cut-off rule with s^* is cut-off in stationary distribution)*[5]

$$x^*(B) = m\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) \, x^*(ds).$$

4. *Market clearing:*

$$Y^d(p^\star) = \int_{s^*}^{\bar{s}} s f(n^\star(s, p^\star)) dx^\star(ds)$$

[5] If we do not have such cut-off rule we have to define

$$x^*(B) = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{s' \in B\}} \mathbf{1}_{\{d(s', p^*)=1\}} x^*(ds) + \mu^* \int_S \mathbf{1}_{\{s \in B\}} \mathbf{1}_{\{d(s, p^*)=1\}} \gamma(ds)$$

where

$$\mu^* = \int_S \sum_{s' \in S} \Gamma_{ss'} \mathbf{1}_{\{d(s', p^*)=0\}} x^*(ds)$$

You can think of condition (2) as a "*no money left over the table*" condition, which ensures additional entrants find unprofitable to participate in the industry.

We can use this model to compute interesting statistics. For example the average output of the firm is given by

$$\frac{Y}{N} = \frac{\sum sf(n^*(s))x^*(ds)}{\sum x^*(ds)}$$

Next, suppose that we want to compute the share of output produced by the top 1% of firms. To do this we first need to compute $\widetilde{s}$ such that

$$\frac{\sum_{\widetilde{s}}^{\bar{s}} x^*(ds)}{N} = .01$$

where N is the total measure of firms. Then the share output produced by these firms is given by

$$\frac{\sum_{\widetilde{s}}^{\bar{s}} sf(n^*(s))x^*(ds)}{\sum_{\underline{s}}^{\bar{s}} sf(n^*(s))x^*(ds)}$$

Suppose now that we want to compute the fraction of firms who are in the top 1% two periods in a row. This is given by

$$\sum_{s \geq \widetilde{s}} \sum_{s' \geq \widetilde{s}} \Gamma_{ss'} x^*(ds)$$

We can use this model to compute a variety of other statistics including the Gini coefficient.

2.5 Adjustment Costs

To end with this section it is useful to think about environments in which firm's productive decisions are no longer static. A simple way of introducing dynamics is by adding adjustment costs.

We will consider labor adjustment costs.[6] Consider a firm that enters period t with n_{t-1} units of labor, hired in the previous period. We can consider three specifications for the adjustment costs, due to hiring n_t units of labor in t, $c(n_t, n_{t-1})$:

- *Convex Adjustment Costs*: if the firm wants to vary the units of labor, it has to pay $\alpha (n_t - n_{t-1})^2$ units of the numeraire good. The cost here depends on the size of the adjustment.

- *Training Costs or Hiring Costs*: if the firm wants to increase labor, it has to pay $\alpha [n_t - (1-\delta) n_{t-1}]^2$ units of the numeraire good, only if $n_t > n_{t-1}$; we can write this as

$$\mathbf{1}_{\{n_t > n_{t-1}\}} \alpha [n_t - (1-\delta) n_{t-1}]^2,$$

  where $\mathbf{1}$ is the indicator function, and δ measures the exogenous attrition of workers in each period.

- *Firing Costs.*

The recursive formulation of the firm's problem would be:

$$V(s, n_-, p) = \max \left\{ 0, \max_{n \geq 0} \; sf(n) - wn - c^v - c(n, n_-) + \frac{1}{(1+r)} \sum_{s' \in S} \Gamma_{ss'} V(s', n, p) \right\}, \quad (6)$$

where c gives the specified cost of adjusting n_- to n. Note that due to limited liability of the firm, the exit value of the firm is 0 and not $-c(0, n_-)$.

Now, a firm is characterized by both its productivity s and labor n_- in the previous period. Note that since the production function f has decreasing returns to scale, there exists an amount of labor $\bar{N}$ such that none of the firms hire labor greater than $\bar{N}$. So, $n_- \in N := [0, \bar{N}]$. Let $\mathcal{N}$ be a σ-algebra on N. If the labor policy function is $n = g(s, n_-)$, then the law of motion now becomes:

$$x'\left(B^S, B^N\right) = m\gamma \left(B^S \cap [s^*, \bar{s}]\right) \mathbf{1}_{\{0 \in B^N\}} + \int_{s^*}^{\bar{s}} \int_0^{\bar{N}} \mathbf{1}_{\{g(s,n_-) \in B^N\}} \Gamma\left(s, B^S \cap [s^*, \bar{s}]\right) x(ds, dn_-),$$

[6] These costs work pretty much like capital adjustment costs, as one might suspect.

$$\forall B^S \in \mathcal{S}, \quad \forall B^N \in \mathcal{N}.$$

Exercise 5 *Write the first order conditions for the problem in* (6)*.*

Define the recursive competitive equilibrium for this economy.

Exercise 6 *Another example of labor adjustment costs is when the firm has to post vacancies to attract labor. As an example of such case, suppose the firm faces a firing cost according to the function c. The firm also pays a cost κ to post vacancies, and after posting vacancies, it takes one period for the workers to be hired. How can we write the problem of firms in this environment?*

2.6 Non-stationary Equilibrium

Up until now we focus on the *stationary industrial equilibrium*, in which individual firms enter and exit, but the whole distribution of firms stays invariant. A more interesting case is to look at the *non-stationary equilibrium* and examine how the distribution of firms shift across time.

Let's maintain our baseline model (with entry & exit, but no adjustment costs), and think about the economy starting with some (arbitrary) initial distribution of incumbent firms x_0. We can imagine that, without any shocks, the firm distribution would converge to the stationary equilibrium distribution x^* defined in 2.4. And on the transitional path towards the stationary equilibrium, firms would face a sequence of prices $\{p_t\}_{t=0}^{\infty}$. We now feed in shocks. We will maintain that wage is normalized to 1 and prices p_t each period is going to be pinned down by equating the endogenous aggregate supply and ad-hoc aggregate demand which we denote $D(p_t, z_t)$, where z_t is a demand side shock that shifts aggregate demand.

It's important that we make it clear on the nature of this shock. In general, z_t can be *deterministic* or *stochastic*. Deterministic shocks are fully anticipated by agents in the economy. Stochastic shocks, on the other hand, come in a random manner and agents only know the random process that governs them. Solving the model with deterministic shocks are not harder than solving the transitional path of the model with no shocks. But models with stochastic shocks are much harder to solve. We will say for now the z_t shocks are deterministic and thus focus on the notion of perfect foresight equilibrium (PFE).

We are now ready to define the firm's problem. Note now state variables would incorporate both the individual state s (idiosyncratic productivity shock) and aggregate states: z (aggregate demand shock) and x (measure of firms).

$$V(s, z_t, x_t) = \max \left\{ 0, \pi(s, z_t, x_t) + \frac{1}{1+r} \sum_{s'} \Gamma_{ss'} V(s', z_{t+1}, x_{t+1}) \right\} \tag{7}$$
$$s.t. \quad \pi(s, z_t, x_t) = \max_{n \geq 0} \; p_t(z_t, x_t) s f(n) - w n_t - c^v$$

Note that we can maintain the cutoff property of the decision rule given our regularity conditions. Let's denote the exit cutoff s_t^*. Note that in order to solve the problem, firms need to know the measure of firms. So we need to figure out the law of motion of firm measure. For each $B \in \mathcal{S}$, we should have

$$x_{t+1}(B) = m_{t+1}\gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}]) x_t(ds) \tag{8}$$

where m_{t+1} is the mass of firms that enter at the beginning of period $t+1$, which is pinned down by the free entry condition

$$\int V(s, z_t, x_t)\gamma(ds) \leq c^e \tag{9}$$

with strict equality holds if $m_t > 0$. The distribution of the initial draw γ and entry cost c^e are exogenously given. Finally, the market clearing condition will close the model by pinning down price p_t

$$D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} s p_t f(n^*(s, z_t, x_t)) x_t(ds) \tag{10}$$

Exercise 7 *Figure out the time line behind the above formulation of the firm's problem, the law of motion of firm measure, and the free entry condition.*

We can thus define the perfect foresight equilibrium as following

Definition 10 *For a given path of shock realizations* $\{z_t\}$ *and a initial firm measure* x_0*, a perfect foresight equilibrium (PFE) for this environment consists of sequences of functions* $\{p_t, m_t, s_t^*, x_t\}$*, that satisfy:*

1. **Optimality**: *given* $\{p_t\}$, $\{s_t^*\}$ *solve the firm's problem* (7) *for each period t.*
2. **Free entry**: $\int V(s, z_t, x_t)\gamma(ds) \leq c^e$*, with strict equality holds if* $m_t > 0$*.*
3. **Law of motion**: $x_{t+1}(B) = m_{t+1}\gamma(B \cap [s_{t+1}^*, \bar{s}]) + \int_{s_t^*}^{\bar{s}} \Gamma(s, B \cap [s_{t+1}^*, \bar{s}])x_t(ds)$, $\forall B \in \mathcal{S}$.
4. **Market clearing**: $D(p_t, z_t) = \int_{s_t^*}^{\bar{s}} sp_t f(n^*(s, z_t, x_t))x_t(ds)$.

Having figured out the equilibrium of the perfect foresight model, the natural next step is thus to solve the fully stochastic equilibrium. It is actually a much harder one. We will resort to some notion of linearization to achieve that. So we will divert a bit in the next subsection to talk about linear approximation.

2.7 Digression: Linear Approximation

To better understand the linearization, let's look at a very basic growth model and approximate the solution linearly. Consider such a social planner's problem (with full depreciation)

$$\begin{aligned} v(k_t) = &\max_{c_t, k_{t+1}} u(c_t) + \beta v(k_{t+1}) \\ &\text{s.t. } c_t + k_{t+1} \leq f(k_t), \ \ \forall t \geq 0 \\ &c_t, k_{t+1} \geq 0, \ \ \forall t \geq 0 \\ &k_0 > 0 \text{ given.} \end{aligned} \tag{11}$$

We can show that $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is a solution to the above social planner's problem if and only if

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1}), \forall t \geq 0 \tag{12}$$

$$c_t + k_{t+1} = f(k_t), \forall t \geq 0 \tag{13}$$

$$\lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{14}$$

Exercise 8 *Prove the above claim.*

We will focus on cases where a steady state k^* exists. Note that the above necessary and sufficient condition give us a second order difference equation system (we can combine the above solution as $\psi(k_t, k_{t+1}, k_{t+2}) = 0$), with exactly two boundary conditions. so the model is totally solvable. The question is how to do that. One obvious option is to find the global solution. For instance, you can guess a k_1, use k_0 and $\psi(k_t, k_{t+1}, k_{t+2}) = 0$ to get $k_2, k_3, \ldots$ forward up until some k_T, and adjust k_1 to make sure k_T is close enough to the steady state k^* (this is called forward shooting). Or you can guess a k_{T-1} and do it backward (which is called backward shooting). Or you can guess and adjust the whole path (which is called the extended path method). All these methods will give you a numerical solution starting from an arbitrary k_0 (that's why we call it a *global* solution).

You can see the above process is time consuming. Linearization is a short cut, that can yield good approximation of the solution *locally*, that is, around the neighborhood of some point. Usually, people will do it around the steady state. The idea is simple. We know the true solution is in the form of $k_{t+1} = g(k_t)$. Let's simply use a linear function to approximate the true solution $g(\cdot)$. Let's say our approximation is $k_{t+1} = \hat{g}(k_t) = a + bk_t$. Then we only need to figure two numbers: a and b. We thus need two conditions. Since we know the steady state is k^*, which means $a + bk^* = k^*$, we get one condition for free (remember we approximate around k^*). Where to find the other one?

We can only find it in ψ and our criteria is that we are going to choose b such that the slope of $\hat{g}$ exactly matches the slope of true decision rule g *at the steady state* k^*. So we take a first order Taylor expansion of $\psi[k, g(k), g(g(k))]$ around k^* and obtain

$$\psi[k, g(k), g(g(k))] \approx \psi(k^*, k^*, k^*) + \psi_k(k^*, k^*, k^*)(k - k^*) \tag{15}$$

We know $\psi[k, g(k), g(g(k))] = 0$, and k is in the neighborhood of k^*, so it must be

$$\psi_k(k^*, k^*, k^*) = \psi_1^* + \psi_2^* g'(k^*) + \psi_3^* g'(k^*) g'(k^*) = 0 \tag{16}$$

Solve this equation gives us $g'(k^*)$ which is exactly what we need (note ψ_1, ψ_2, and ψ_3 may also involve $g'(k^*)$). We can then let $b = g'(k^*)$ and use $\hat{g}(k_t) = a + bk_t$ to approximate the solution near the steady state.

Comment 2 *In practice, it's messy to do the total derivative as above. A cleaner way is to linearize the system directly with k_t, k_{t+1}, k_{t+2}, and then solve the linear system using whatever method you like. Usually, we cast it on a state space and solve it using matrix algebra (here it helps to know some econometrics).*

Exercise 9 *Suppose $f(k_t) = k_t^\alpha$, $u(c_t) = \ln c_t$. Verify that the solution to the social planner's problem is $k_{t+1} = \alpha\beta k_t^\alpha$. Get the linearized solution around the steady state and compare it with the closed form solution. How precise is the linear approximation?*

Exercise 10 *Extend the linearization to the case where we have stochastic productivity shocks z_t.*

3 Incomplete Market Models

3.1 A Farmer's Problem

Consider the following problem of a farmer:

$$V(s,a) = \max_{c,a'} \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \tag{17}$$

$$\begin{aligned} s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq 0, \end{aligned}$$

where a is his holding of coconuts, which can only take positive values, c is his consumption, and s is amount of coconuts that he produces. s follows a Markov chain, taking values in a finite set S. q is the fraction of coconuts that can be stored to be consumed tomorrow. Note that, the constraint on the holding of coconuts tomorrow, is a constraint imposed by nature; nature allows the farmer to store coconuts at rate $1/q$, but, it does not allow him to *transfer coconuts* from tomorrow to today.

We are going to consider this problem in the context of a partial equilibrium, where q is given. The first question is, what can be said about q?

Remark 2 *Assume there are no shocks in the economy, so that s is a fixed number. Then, we could write the problem of the farmer as:*

$$V(a) = \max_{c,a' \geq 0} \{u(a + s - qa') + \beta V(a')\}.$$

If u is assumed to be logarithmic, the first order condition for this problem implies:

$$\frac{c'}{c} \geq \frac{\beta}{q},$$

with equality if $a' > 0$. Therefore, if $q > \beta$ (i.e. nature is more stingy than farmer's patience), then $c' < c$, and the farmer dis-saves (at least, as long as $a' > 0$). But, when $q < \beta$, consumption grows without bound. This is the reason we put this assumption on the model, in what follows.

A crucial assumption for generating a bounded asset space is $\beta/q < 1$, stating that agents are sufficiently impatient so they tend to consume more and decumulate their asset when they get richer and far away from the non-negativity constraint, $a' \geq 0$. However, this does not mean that, when faced with a possibility of very low consumption, agents would not save, even though the rate of return, $1/q$, is smaller than the rate of impatience $1/\beta$.

The first order condition for farmer's problem (17) is given by:

$$u_c\left(c\left(s,a\right)\right) \geq \frac{\beta}{q}\sum_{s'}\Gamma_{ss'}u_c\left(c\left(s',a'\left(s,a\right)\right)\right),$$

with equality when $a'\left(s,a\right) > 0$, where $c\left(\cdot\right)$ and $a'\left(\cdot\right)$ are policy functions from the farmer's problem. Notice that $a'\left(s,a\right) = 0$ is possible, if we assume appropriate shock structure in the economy. Specifically, it depends on the value of $s_{\min} := \min_{s_i \in S}\ s_i$.

The solution to the problem of the farmer, for a given value of q, implies a distribution of coconut holdings in each period. This distribution, together with the Markov chain describing the evolution of s, can be summed together as a single probability measure for the distribution of shocks and assets (coconut holdings) over the product space $E = S \times \mathbb{R}_+$, and its σ-algebra, $\mathcal{B}$, which we denote by X. The evolution of this probability measure is given by:

$$X'\left(B\right) = \sum_{s' \in B_s}\int \Gamma_{ss'}\mathbf{1}_{\{a'(s,a)\in B_a\}}dX\left(s,a\right), \quad \forall B \in \mathcal{B}, \tag{18}$$

where B_s and B_a are the S-section and $\mathbb{R}$-section of B (projections of B on S and $\mathbb{R}_+$), respectively, and $\mathbf{1}$ is the indicator function. Let $\widetilde{T}\left(\Gamma, a', \cdot\right)$ be the mapping associated with (18) (the adjoint operator), so that:

$$X'(B) = \widetilde{T}(\Gamma, a', X)(B), \quad \forall B \in \mathcal{B}.$$

Define $\widetilde{T}^n(\Gamma, a', \cdot)$ as:

$$\widetilde{T}^n(\Gamma, a', X) = \widetilde{T}\left(\Gamma, a', \widetilde{T}^{n-1}(\Gamma, a', X)\right).$$

Then, we have the following theorem.

Theorem 1 *Under some conditions on $\widetilde{T}(\Gamma, a', \cdot)$, there is a unique probability measure X^*, so that:*

$$X^*(B) = \lim_{n\to\infty} \widetilde{T}^n(\Gamma, a', X_0)(B), \quad \forall B \in \mathcal{B},$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

A condition that makes things considerably easier for this theorem to hold is that E is a compact set. Then, we can use Theorem (12.12) in **?**, to show this result holds. Given that S is finite, this is equivalent to a compact support for the distribution of asset holdings. We discuss this in detail in Appendix **??**.

3.2 Huggett Economy

Now we modify the farmer's problem in (17) a little bit (**?**):

$$\begin{aligned} V(s, a) = \max_{c, a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s', a') \right\} \qquad (19) \\ s.t. \quad & c + qa' = a + s \\ & c \geq 0 \\ & a' \geq \underline{a}, \end{aligned}$$

where $\underline{a} < 0$, so now farmers can borrow and lend with each other, but with a borrowing limit. We continue to make the same assumption on q; i.e. $\beta/q < 1$. Solving this problem gives the policy function $a'(s,a)$. It is easy to extend the analysis in the last section to show that there is an upper bound of the asset space, which we denote by $\bar{a}$, so that for any $a \in A := [\underline{a}, \bar{a}]$, $a'(s,a) \in A$, for any $s \in S$.

Remark 3 *One possibility for $\underline{a}$ is what we call the natural borrowing constraint. This is the constraint that ensures the agent can pay back his debt for sure, no matter what the nature reveals (whatever sequence of idiosyncratic shocks is realized). This is given by*

$$a^n := -\frac{s_{\min}}{\left(\frac{1}{q} - 1\right)}.$$

If we impose this constraint on (19)*, even when the farmer receives an infinite sequence of bad shocks, he can pay back his debt by setting his consumption equal to zero, forever.*

But, what makes this problem more interesting is to tighten this borrowing constraint; the natural borrowing constraint is very unlikely to be hit. One way to restrict borrowing is no borrowing at all, as in the previous section. Another case is to choose $0 > \underline{a} > a^n$. This is the case that we consider in this section.

Now suppose there is a (unit) mass of farmers with distribution function $X(\cdot)$, where $X(D,B)$ denotes fraction of people with shock $s \in D$ and $a \in B$, where D in an element of the power set of S , $P(S)$ (which, when S is finite, is the natural σ-algebra over S) and B is a Borel subset of A, $B \in \mathcal{A}$. Then the distribution of farmers tomorrow is given by:

$$X'(S', B') = \int_{A \times S} \mathbf{1}_{\{a'(s,a) \in B'\}} \sum_{s' \in S'} \Gamma_{ss'} dX(s,a), \tag{20}$$

for any $S' \in P(S)$ and $B' \in \mathcal{A}$.

Implicitly this defines an operator T such that $T(X) = X'$. If T is *sufficiently nice*, then there exits a unique X^* such that $X^* = T(X^*)$ and $X^* = \lim_{n\to\infty} T^n(X_0)$, for any initial distribution over the product space $S \times A$, X_0. Note that the decision rule is obtained given q. Hence, the resulting stationary distribution X^* also depends on q. So, let us denote it by $X^*(q)$.

To determine the equilibrium value of q, in a general equilibrium setting, consider the following variable (as a function of q):

$$\int_{A \times S} a dX^*(q).$$

This expression give us the average asset holdings, given the price, q.

We want to determine an endogenous q, so that the asset market clears; we assume that there is no storage technology so that asset supply is 0 in equilibrium. Hence, price q should be such that asset demand equals asset supply; i.e.

$$\int_{A \times S} a dX^*(q) = 0.$$

In this sense, equilibrium price q is the price that generates the stationary distribution that clears the asset market.

We can show that a solution exists by invoking intermediate value theorem, by showing that the following three conditions are satisfied (note that $q \in [\beta, \infty]$):

1. $\int_{A \times S} a dX^*(q)$ is a continuous function of q;
2. $\lim_{q \to \beta} \int_{A \times S} a dX^*(q) \to \infty$; (Intuitively, as $q \to \beta$, interest rate increases. Hence, agents would like to save more. Together with precautionary savings motive, they accumulate unbounded quantities of assets in the stationary equilibrium, in this case.) and,
3. $\lim_{q \to \infty} \int_{A \times S} a dX^*(q) < 0$. (This is also intuitive; as $q \to \infty$, interest rate converges to 0. Hence, everyone would rather borrow.)

3.3 Aiyagari Economy

In the **?** Economy, there is physical capital. In this sense, the average asset holdings in the economy must be equal to the average (physical) capital. So, if we keep denoting the stationary distribution of assets by X^*, we must have:

$$\int_{A\times S} adX^*(q) = K,$$

where A is the support of the distribution of wealth. (It is not difficult to see that this set is compact.)

On the other hand, the shocks affect the labor income; we can think of these shocks as fluctuations in the employment status of individuals. Thus, the problem of an individual in this economy can be written as:

$$\begin{aligned} V(s,a) = \max_{c,a'} \quad & \left\{ u(c) + \beta \sum_{s'} \Gamma_{ss'} V(s',a') \right\} \\ s.t. \quad & c + a' = (1+r)a + ws \\ & c \geq 0 \\ & a' \geq \underline{a}, \end{aligned} \tag{21}$$

where, here, r is the return to savings, and w is the wage rate. Therefore,

$$\int_{A\times S} sdX^*(q)$$

gives the average labor in this economy. If we decide to think of agents supplying one unit of labor, then, we may think of the expression as determining the *effective labor supply*.

We assume the standard constant returns to scale production technology for the firm, of the form:

$$F(K,L) = AK^{1-\alpha}L^{\alpha},$$

with the rate of depreciation δ. Hence:

$$\begin{aligned} r =& F_k(K, L) - \delta \\ =& (1-\alpha) A \left(\frac{K}{L}\right)^{-\alpha} - \delta \\ =:& r\left(\frac{K}{L}\right), \end{aligned}$$

and

$$\begin{aligned} w =& F_l(K, L) \\ =& \alpha A \left(\frac{K}{L}\right)^{1-\alpha} \\ =:& w\left(\frac{K}{L}\right). \end{aligned}$$

In terms of Huggett **?**, q, the price of assets, is given by

$$q = \frac{1}{(1+r)} = \frac{1}{[1 + F_k(K, L) - \delta]}.$$

Therefore, prices faced by agents are all functions of the capital-labor ratio. As a result, we may write the stationary distribution of assets as a function of capital-labor ratio as well, $X^*\left(\frac{K}{L}\right)$. Thus, the equilibrium condition becomes:

$$\frac{K}{L} = \frac{\int_{A\times S} a dX^*\left(\frac{K}{L}\right)}{\int_{A\times S} s dX^*\left(\frac{K}{L}\right)}.$$

Using this condition, one can solve for the equilibrium capital-labor ratio.

3.3.1 Policy

In **?** or **?** economy, model parameters can be summarized by $\theta = \{u, \beta, s, \Gamma, F\}$. In stationary equilibrium, value function $v(s, a; \theta)$ as well as $X^*(\theta)$ can be obtained, where $X^*(\theta)$ is a mapping from model parameters to stationary distribution of agent's asset holding and shocks. Suppose now there is a policy change that shifts θ to $\hat{\theta} = \{u, \beta, s, \hat{\Gamma}, F\}$. Associated with this new environment there is a new value function $v\left(s, a; \hat{\theta}\right)$ and $X^*\left(\hat{\theta}\right)$. Define $\eta(s,a)$ to be the solution of:

$$v\left(s, a + \eta(s,a), \hat{\theta}\right) = v(s, a, \theta)$$

which is the transfer payment necessary to the households so that they are indifferent between living in the old environment and in the new one. Hence total payment needed to compensate households for this policy change is given by:

$$\int_{A\times S} \eta(s,a)\, dX^*(\theta)$$

Notice that the changes do not take place when the government is trying to compensate the households. Hence we use the original stationary distribution associated with θ to aggregate the households.

If $\int_{A\times S} v(s,a)\, dX^*\left(\hat{\theta}\right) > \int_{A\times S} v(s,a)\, dX^*(\theta)$, does this necessarily mean that households are willing to accept this policy change? The answer is not necessarily because the economy may well spend a long time in the transition path to the new steady state, during which there may be severe welfare loss.

3.3.2 Aggregate Shocks

In this section, we consider an economy that is subject to both aggregate and idiosyncratic shocks, at the same time; consider **?** economy again, now, with a production function that is subject to an aggregate shock; $zF\left(K, \bar{N}\right)$.

Let X be the distribution of types; then the aggregate capital is given by:

$$K = \int a dX(s,a).$$

$$K' = G(z, K)$$

The question is what are the sufficient statistics for predicting the aggregate capital stock and, consequently, prices tomorrow? Are z and K sufficient determine capital tomorrow? The answer to these questions is no, in general; this is true if, and only if, the decision rules are linear. Therefore, X, the distribution of types becomes a state variable (even in the stationary equilibrium) for this economy.

Then, the problem of an individual becomes:

$$
\begin{aligned}
V(z,X,s,a) = \max_{a'} \quad & \left\{ u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V(z',X',s',a') \right\} \\
s.t. \quad & c + a' = azf_k\left(K,\bar{N}\right) + szf_n\left(K,\bar{N}\right) \\
& K = \int a dX(s,a) \\
& X' = G(z,X) \\
& c, a' \geq 0.
\end{aligned}
$$

Computationally, this problem is a beast! So, how can we solve it? To provide some idea, we will first consider an economy with *dumb* agents!

Consider an economy in which people are stupid; people believe tomorrow's capital depends only on K, and not X. This, obviously, is not an economy in which expectations are rational. Nevertheless, people's problem in such settings becomes:

$$
\begin{aligned}
\widetilde{V}(z,X,s,a) = \max_{a'} \quad & \left\{ u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} V(z',X',s',a') \right\} \\
s.t. \quad & c + a' = azf_k\left(K,\bar{N}\right) + szf_n\left(K,\bar{N}\right) \\
& K = \int a dX(s,a) \\
& X' = \widetilde{G}(z,K) \\
& c, a' \geq 0.
\end{aligned}
$$

Next step is to allow people become slightly smarter; they now can use extra information, like mean and variance of X, to predict X'. Does this economy work better than our *dumb benchmark*? Computationally no! This answer, as stupid as it may sound, has an important message: people actually act linearly in the economy; decision rules are approximately linear. Therefore, we may use **?** results without fear; the approximations are quite reliable!

3.3.3 Linear Approximation Revisited

Let's now continue our discussion of linear approximation in the context of Aiyagari model. As we can see in section 3.3.2, solving the heterogeneous agent model with aggregate shocks is computationally hard. We need to guess a reduced form rule for agents to forecast future prices, and when the model has frictions on several dimensions, little could we say on how to choose such a rule.

We can, however, use linear approximation as a short cut to obtain the solution near the steady state. The idea is as following: firstly, starting from the steady state, obtain the perfect foresight equilibrium (PFE) path given a specified path of small deterministic shocks, and then use the PFE to approximate the behavior of the economy facing small stochastic shocks around the steady state. This method is proposed recently by **?**[7].

To fix the idea, let's consider the above Aiyagari model with a TFP shock z. Consider $\log(z_t)$ follows an AR(1) process with a serial correlation parameter ρ. Thus, (the log of) the shock will go up by, say, one unit in period 0 and thus delivers the full sequence of values $(1, \rho, \rho^2, \rho^3, \dots)$. When we solve for our resulting deterministic equilibrium transition path, the individual takes as given a sequence of prices and because it is irrelevant for the individual how these prices are determined, they can be summarized as depending simply on time. Solving the deterministic path is straightforward: we guess on a price path (or the path for an aggregate variable like consumption), solve the household's problem backwards—given that we know that there will be convergence back to the same steady state—and then derive the aggregate implications of the households' behavior and update our guess for the price path. This iterative procedure is also standard and fully nonlinear.

After solving the PFE, we have a sequence of whatever variable we care about. Let's label this sequence:

[7] The description of the method below is from their paper with slight modifications.

$\{x_0, x_1, x_2, \dots\}$. Now consider the same economy subjected to recurring aggregate shocks to z. The key assumption behind this procedure is that we regard the x sequence in response to the one-time shock as well approximated by a linear system. A linear system would precisely mean that the effects of shocks are linearly scalable and additive so that the *level* of x at some future time T, after a sequence of random shocks is given by

$$x_T \approx x_0\epsilon_T + x_1\epsilon_{T-1} + x_2\epsilon_{T-2} + \dots$$

where ϵ_t is the innovation to $\log(z_t)$ at period t. Thus, the model with aggregate shocks can be obtained by mere simulation based on the one deterministic path: the non-linear impulse response function of the PFE.

3.4 Aiyagari Economy with Entrepreneurs

Aiyagari economy now becomes the workhorse of modern macroeconomics. It features incomplete markets and an endogenous wealth distribution, in which we can examine interactions between heterogeneous agents and distributional effects of public polices. Now let's take a quick look on a very simple extension and have a sense on how the model can be used to study a wide variety of macroeconomic issues.

Specifically, we will introduce entrepreneurs into the Aiyagari world. Suppose every period agents choose an occupation: to be an entrepreneur or to be a worker. Entrepreneurs run their own business, and workers supply labor in the market. Entrepreneurs can manage one project which combines her entrepreneur ability (ϵ), capital(k) and labor(n).

Let's denote $V^w(a, s, \epsilon)$ the value of a worker with wealth a, labor productivity s, and entrepreneur ability ϵ. Also denote $V^e(a, s, \epsilon)$ the value of an entrepreneur. The worker's problem is to choose tomorrow's occupation and wealth level, as well as today's consumption, at given wage rate w and interest rate r.

$$
\begin{aligned}
V^w(a, s, \epsilon) = \max_{c,a',i} & u(c) + \beta\{i\mathbb{E}\left[V^w(a', s', \epsilon')\right] + (1-i)\mathbb{E}\left[V^e(a', s', \epsilon')\right]\} \\
s.t. \quad & c + a' = ws + (1+r)a \\
& a' \geq 0
\end{aligned}
$$

Similarly, the entrepreneur's problem can be formulated as following

$$
\begin{aligned}
V^e(a, s, \epsilon) = \max_{c,a',i} & u(c) + \beta\{i\mathbb{E}\left[V^w(a', s', \epsilon')\right] + (1-i)\mathbb{E}\left[V^e(a', s', \epsilon')\right]\} \\
s.t. \quad & c + a' = \pi(a, s, \epsilon) \\
& a' \geq 0
\end{aligned}
$$

Note the entrepreneur's income is from profits $\pi(a, s, \epsilon)$ rather than wage. We assume entrepreneurs have access to a DRS technology f, that can produce output given (k, n, ϵ). After paying costs of factors and loans, profits $\pi(a, s, \epsilon)$ are given by

$$
\begin{aligned}
\pi(a, s, \epsilon) = \max_{k,n} & f(k, n, \epsilon) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s, 0\} \\
s.t. \quad & k - a \leq \phi a
\end{aligned}
$$

The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth. A limit of this model is that entrepreneurs never make an operating loss within a period, as they can always choose $k = n = 0$ and earn the risk free rate on saving. In this model, agents with high entrepreneur ability have access to a investment technology f with higher return than workers, and therefore they accumulate wealth faster.

References