Appendix (for Review and Online Publication Only)

A Exploiting Sequentially Complete Markets in the Economy with Endogenous Portfolios

Since in our applications the number of values the aggregate state z^{prime} can take tomorrow is two for every state z today, markets are sequentially complete when households can freely trade a bond and a stock, even though z can take three values. We exploit this for the purposes of characterizing equilibrium prices numerically. In particular, after having solved for equilibrium allocations (as described in the next section), we can easily construct prices of state-contingent claims (Arrow securities). We then reconstruct the equilibrium prices of conventional stocks and bonds as additional (effectively redundant) assets. For completeness, we here supply the equilibrium definition for the economy with a full set of Arrow securities.

Let $a_i(z')$ be shares of stock purchased by a household of age i. These shares represent a claim to fraction $a_i(z')$ of the capital stock if and only if aggregate state z' is realized in the next period. The state of the economy is the distribution of shares of stock A, given the current period shock z. We denote the state-contingent stock prices $P(z, A, z')$.

With this asset market structure, the maximization problem of the households now reads as

$$v_i(z, A, a) = \max_{c \geq 0, a'(z')} \left\{ u(c) + \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'} \; v_{i+1}\left(z', A'(z'), a'(z')\right) \right\} \tag{A-1}$$

$$\text{s.t.} \quad c + \sum_{z'} \left[a'(z') - a\right] \; P(z, A, z') = \varepsilon_i(z) w(z) + d(z) a \tag{A-2}$$

$$A'(z') = G(z, A, z') \tag{A-3}$$

with solution $c_i(z, A, a)$, $a_i'(z, A, a, z')$.

Definition 2. *A recursive competitive equilibrium with complete markets are value and policy functions* $\{v_i, c_i, a_i'\}$, *pricing functions* w, d, P, *and an aggregate law of motion* G *such that:*

1. *Given the pricing functions and the aggregate law of motion, the value functions* $\{v_i\}$ *solve the recursive problem of the households and* $\{c_i, a_i'\}$ *are the associated policy functions.*

2. *Wages and dividends satisfy*

$$w(z) = (1-\theta)z \qquad \textit{and} \qquad d(z) = \theta z. \tag{A-4}$$

3. *Markets clear*

$$\sum_{i=1}^{I} c_i(z, A, A_i) = z \tag{A-5}$$

$$\sum_{i=1}^{I} a_i'(z, A, A_i, z') = 1 \qquad \forall z' \in Z. \tag{A-6}$$

4. *The aggregate law of motion is consistent with individual optimization*

$$\begin{aligned} G_1(z, A, z') &= 0 \\ G_{i+1}(z, A, z') &= a_i'(z, A, A_i, z') \qquad \forall z',\ i = 1, ..., I-1. \end{aligned} \tag{A-7}$$

We now describe how we reconstruct returns and prices for conventional stocks and bonds, given the prices of state-contingent shares, exploiting the equivalence between the two market structures when the aggregate shock takes only two values. Let $W(z, A)$ denote the value of the (unlevered) firm after it has paid out dividends. This is equal to the price of all state-contingent shares:

$$W(z, A) = \sum_{z' \in Z} P(z, A, z'). \tag{A-8}$$

In the presence of state-contingent shares, risk-free bonds and levered stocks are redundant assets, but they can still be priced. We now compute these prices $q(z, A)$ and $p(z, A)$ as functions of the state-contingent prices $P(z, A, z')$ and $W(z, A)$. There are two ways of securing one unit of the good unconditionally in the next period. One could either buy one unit of the risk-free bond at price $q(z, A)$ or instead buy a bundle of state-contingent shares for each possible z', setting the state-specific quantity to $1/\left[W(z', G(z, A, z')) + \theta z'\right]$ so as to ensure a gross payout of one in each state. A no-arbitrage argument implies that the cost of the two alternative portfolios must be identical:

$$q(z, A) = \sum_{z'} \frac{P(z, A, z')}{W(z', G(z, A, z')) + \theta z'}. \tag{A-9}$$

With the bond price in hand, the stock price can immediately be recovered from the condition that the value of the unlevered firm (in the economy with state-contingent shares) must equal the value of levered stocks and risk-free bonds:

$$p(z, A) = W(z, A) - q(z, A)B. \tag{A-10}$$

B Computational Appendix

Even for a moderate number of generations, the state space is large: $I-2$ continuous state variables (plus z). Since we want to deal with large shocks, local methods should be used with caution. We therefore use global approximation on sparse grids, thereby respecting the size of our aggregate shock while avoiding the curse of dimensionality. The baseline model takes advantage of the fact that our two assets span the space of possible shocks, therefore allowing us to solve the model via a planning problem using the Negishi algorithm. The fixed-portfolio economies do not solve such a planning problem, so we must directly solve for the competitive equilibrium.

B.1 Endogenous Portfolio Economy

In our economy with endogenous trade in stocks and bonds, there are two assets and two values for the aggregate shock. Thus, the set of agents who are active at dates $t-1$ and t select portfolios that pool date t risk perfectly and share the same growth rate for the marginal utility of consumption. But it is impossible for agents active at date $t-1$ to share risk with agents who enter the economy at date t, and thus shocks at t reallocate resources between the newborn and existing cohorts.

The computational challenge for characterizing equilibrium allocations is to characterize this reallocation. As Brumm and Kubler (2013) show, the key condition that pins down the share of the newborns is their lifetime budget constraint that must be satisfied. Thus, computing a competitive equilibrium amounts to solving for a law of motion for the consumption share of newborn agents, with the property that the present values of lifetime income and consumption are equal with zero initial wealth. We now formally describe this way of characterizing competitive equilibrium.

Let the aggregate state be z and the vector $\lambda = \lambda_1, ..., \lambda_I$ where $\lambda_i \in [0, 1]$ for all i. Let λ define age group i's share of aggregate output as follows for $i = 1, ..., I$:

$$c_i(z, \lambda) = \lambda_i z \tag{A-11}$$

Let $\lambda' = G(z', \lambda)$ define a law of motion for λ and thus a resource-feasible allocation. We will use the notation $\lambda'_i = G_i(z', \lambda)$.

Allocations in the competitive equilibrium with endogenous portfolios are defined by one particular specification for $G(z', \lambda)$.

The numerical challenge is to characterize $G_1(z', \lambda)$ which defines λ'_1. Given $G_1(z', \lambda)$, we will see that the remaining $G_i(z', \lambda)$, for $i = 2, ..., I$, defining consumption shares for agents of ages $i = 2, ..., I$ in the next period are given by

$$\lambda'_i = G_i(z', \lambda) = \beta_i^{\frac{1}{\gamma}} \frac{(1 - G_1(z', \lambda))}{\sum_{i=1}^{I-1} \beta_{i+1}^{\frac{1}{\gamma}} \lambda_i} \times \lambda_{i-1} \qquad i = 2, ..., I. \tag{A-12}$$

The logic for this specification, as we will see shortly, is that it guarantees that all agents share the same state-contingent inter-temporal marginal rate of substitution. Note that, by construction

$$\sum_{i=1}^{I} \lambda'_i = 1$$

We now describe how we solve for the function $G_1(z', \lambda)$ corresponding to the competitive equilibrium.

Define, for $i = 1, ..., I - 1$

$$\begin{aligned} p_i(z, z', \lambda) &= \pi(z'|z) \frac{\Pi_{l=1}^{i+1} \beta_l c_{i+1}(z', \lambda')^{-\gamma}}{\Pi_{l=1}^{i} \beta_l c_i(z, \lambda)^{-\gamma}} \\ &= \pi(z'|z) \frac{\Pi_{l=1}^{i+1} \beta_l \left(\lambda'_{i+1} z'\right)^{-\gamma}}{\Pi_{l=1}^{i} \beta_l \left(\lambda_i z\right)^{-\gamma}} \\ &= \pi(z'|z) \frac{(1 - G_1(z', \lambda))}{\sum_{i=1}^{I-1} \beta_{i+1}^{\frac{1}{\gamma}} \lambda_i} \left(\frac{z'}{z}\right)^{-\gamma}. \end{aligned} \tag{A-13}$$

where the second line substitutes in the consumption sharing rule (A-11), and the third line uses the law of motion (A-12). Note from the third line that $p_i(z, z', \lambda)$ is independent of i.

Next define functions $B_i(z, \lambda)$ as follows, starting from $i = I$, and moving sequentially down to $i = 1$:

$$\begin{aligned} B_I(z,\lambda) &= c_I(z,\lambda) - w_I(z) \\ B_{i-1}(z,\lambda) &= c_{i-1}(z,\lambda) - w_{i-1}(z) + \sum_{z'\in\mathcal{Z}} p(z,z',\lambda) B_i(z',\lambda') \\ B_1(z,\lambda) &= c_1(z,\lambda) - w_1(z) + \sum_{z'\in\mathcal{Z}} p(z,z',\lambda) B_2(z',\lambda'), \end{aligned} \tag{A-14}$$

where in each case $\lambda' = G(z',\lambda)$.

Claim Given the sharing rule (A-11) and the law of motion $G_i(z',\lambda)$ for $i \geq 2$ (A-12), the allocation defined by a function $G_1(z',\lambda)$ is a competitive equilibrium in the stock economy if and only if the implied $B_1(z,\lambda) = 0$ for all z, λ.

Proof First, note that allocations in the stock economy are identical to those in an economy in which agents trade two Arrow securities, each of which pays out if and only if one particular value for the aggregate shock z is realized. In an economy with trade in Arrow securities, the conditions defining a competitive equilibrium are: (i) security prices reflect state-contingent marginal rates of substitution, (ii) the agent's budget constraints are satisfied at each age, where financial wealth at age $i = 1$ is zero, and (iii) the aggregate resource constraint is satisfied. These conditions are all satisfied by the allocation described above: condition (i) is equation (A-13), condition (ii) is equations (A-14), and condition (iii) is satisfied by virtue of equation (A-11).

Note finally that for computational purposes it is not necessary to carry around the entire vector λ since $\sum_{i=1}^{I} \lambda_i = 1$. Thus, a sufficient state vector is $\left(z, \{\lambda_i\}_{i=1}^{I-1}\right)$. The law of motion A-12 is still sufficient to define consumption for all age groups in the next period.

We now move to solving for the unknown function G_1 described above. In order to solve the model we implement the following algorithm:

1. Initiate a grid of $\mathcal{L} = \{\lambda^j\}_{j=1}^{J}$, where each λ^j is an $I-1$ dimensional vector. These will be the collocation points and are in practice sets on a Smolyak grid.

2. For each $z \in \mathcal{Z}$, and each $j = 1, ..., J$, guess a value $G_1(z, \lambda_j)$. Use these guesses to construct an interpolating function $\hat{G}_1(z,\lambda)$ for any vector λ. In practice, we use Chebyshev polynomials in this step.

3. For the $(\#\mathcal{Z})^{I-1}$ possible histories through which a newborn agent could live, and for each

$j = 1, ..., J$, use $\hat{G}_1$ to construct consumption allocations, Arrow securities prices, and budget errors as described above. The interpolation is necessary because the vector of weights will typically not lie on the grid after one period passes.

4. Steps [2] and [3] create $\#\mathcal{Z} \times J$ equations (the budget errors for each shock value and each collocation vector) in the same number of unknowns (the values of G_1 for each shock and collocation vector). We use a nonlinear root finder to solve this system of equations.

B.2 Fixed Portfolio Economy

Relative to the methods described in Krueger and Kubler (2004, 2006), there are two additional complications in the present model. The first is that, while the sparse grids used there are subsets of $(I-1)-$dimensional cubes, wealth shares used in this paper are defined on the $(I-2)$ dimensional *simplex*. We deal with this issue by defining the state space in levels of wealth rather than in shares, and then we map a generation's level of wealth into a share when evaluating the Euler equations. The second complication is that the prices of the assets cannot be read off the first-order conditions in this model[52] but must instead adjust so that the excess demand for stocks and bonds is zero in both cases. We now describe our algorithm for solving the model.

1 Solve for the steady state prices and wealth *levels*: $\overline{p}, \overline{q}, \overline{W} = (\overline{W}_2, \overline{W}_3, ..., \overline{W}_{I-1}, \overline{W}_I)$. As described above, we work with an endogenous state space of dimension $I-1$ rather than $I-2$ and then map wealth levels into wealth shares.

2 Create a sparse grid around the steady state wealth distribution. Call this grid $\mathbb{W}$. We verify ex-post that the wealth distribution stays within this hyper-cube along the simulation path.

3 We start with an outer loop over prices (this loop was unnecessary in Krueger and Kubler (2004)). At an outer loop iteration n we have guesses from the previous iteration for Chebyshev coefficients $(\alpha_z^{p,n}, \alpha_z^{q,n})$ for the prices that are used to compute the values of prices (p, q) for each realization of z and each point $W \in \mathbb{W}$ in the endogenous state space. We denote the vector of price values by $(\psi_{z,W}^{p,n}, \psi_{z,W}^{q,n})_{W \in \mathbb{W}}$. The Chebyshev coefficients $(\alpha_z^{p,n}, \alpha_z^{q,n})$ also determine the pricing functions on the entire state space, denoted by $(\widehat{\psi}_z^{p,n}, \widehat{\psi}_z^{q,n})$, somewhat abusing notation.[53]

[52] One can do this in production economies where factor prices equal marginal productivities

[53] Note that this notation implies $\widehat{\psi}_z^{p,n}(W) = \psi_{z,W}^{p,n}$.

4 Given approximate pricing functions in the inner loop, we iterate over household policies. In this loop we generate both the savings policy function and the law of motion for the wealth distribution consistent with approximate price functions $(\widehat{\psi}_z^{p,n}, \widehat{\psi}_z^{q,n})$. The savings policy is indexed by generation and *current* state z, and so the current guess of the savings policy function at policy iteration m when the price iteration is n is determined by Chebyshev coefficients of the form $(\alpha_{z,i}^{y,n,m})$. These can be used to compute the optimal savings level at each grid point W and is denoted by $(\psi_{z,i,W}^{y,n,m})$. As in the previous step, the Chebyshev coefficients also determine the entire approximating savings functions $(\widehat{\psi}_{z,i}^{y,n,m})$. The law of motion for wealth is a function of savings, current prices, and future prices; it must therefore be indexed by current state z, generation i, and future state z'. Similarly, the Chebyshev coefficients $(\alpha_{z,i,z'}^{G,n,m})$ are used to compute the law of motion $(\psi_{z,i,z',W}^{G,n,m})$ for all points $W \in \mathbb{W}$ and to generate the approximating functions $\widehat{\psi}_{z,i,z'}^{G,n,m}$.

5 At this point we loop over each value of z and each point in $W \in \mathbb{W}$ and solve the $I-1$ Euler equations for the $I-1$ optimal savings levels, $y_{i,z,W}$. The Euler equations that we solve to generate the updated savings levels are:

$$u'(c_i(y_{i,z,W}; W, z)) = \beta_i \mathbb{E}_z \widehat{R}_i^{n,m}(z') u'(\widehat{c}_{i+1}(W_+(z'), z'^{n,m}),$$

where

$$\widehat{R}_i^{n,m}(z') = \left(\lambda_i \frac{\widehat{\psi}_{z'}^{p,n}(\psi_{z,z',W}^{G,n,m}) + \theta z' + \overline{B}\widehat{\psi}_{z'}^{q,n}(\psi_{z,z',W}^{G,n,m}) - \overline{B}}{\psi_{z,W}^{p,n}} + \frac{(1-\lambda_i)}{\psi_{z,W}^{q,n}} \right)$$

$$W_+(z') = \begin{bmatrix} W_{+,2}(z') \\ \dots \\ W_{+,I} \end{bmatrix} = \begin{bmatrix} \left(\lambda_1 \frac{\widehat{\psi}_{z'}^{p,n}(\psi_{z,z',W}^{G,n,m}) + \theta z' + \overline{B}\widehat{\psi}_{z'}^{q,n}(\psi_{z,z',W}^{G,n,m}) - \overline{B}}{\psi_{z,W}^{p,n}} + \frac{(1-\lambda_1)}{\psi_{z,W}^{q,n}} \right) y_{1,z,W} \\ \dots \\ \left(\lambda_{I-1} \frac{\widehat{\psi}_{z'}^{p,n}(\psi_{z,z',W}^{G,n,m}) + \theta z' + \overline{B}\widehat{\psi}_{z'}^{q,n}(\psi_{z,z',W}^{G,n,m}) - \overline{B}}{\psi_{z,W}^{p,n}} + \frac{(1-\lambda_{I-1})}{\psi_{z,W}^{q,n}} \right) y_{I-1,z,W} \end{bmatrix}$$

$$
\begin{aligned}
c_1(y_{1,z}; W, z) &= (1-\theta)z\epsilon_1(z) - y_{1,z,W} \\
& \quad \text{for } i = 1, ..., I-1: \\
c_i(y_{i,z}; W, z) &= (1-\theta)z\epsilon_i(z) + \left(\widehat{\psi}^{p,n}_{z,W} + \theta z + \overline{B}\ \widehat{\psi}^{q,n}_{z,W}\right) \frac{W_i}{\sum_{l=2}^{I} W_l} - y_{i,z,W} \\
& \quad \text{for } i = 1, ..., I-2: \\
\widehat{c}_{i+1}(W_+(z'), z'^{n,m}) &= (1-\theta)z\epsilon_{i+1}(z') + W_{+,i+1}(z') - \widehat{\psi}^{y,n,m}_{z',i+1}(W_+(z')) \\
\widehat{c}_I(W_+(z'), z'^{n,m}) &= (1-\theta)z\epsilon_I(z') + W_{+,I}(z').
\end{aligned}
$$

Note that in the calculation of the c_i's, we switch from using wealth levels to using wealth shares to satisfy the requirement that only the latter are truly minimal state variables. This is another difference relative to the previous use of Smolyak polynomials in Krueger and Kubler (2004).

With the new savings in hand, we update the savings policies as $\psi^{y,n,m+1}_{z,i,W} = y_{i,z,W}$. The law of motion for wealth levels is updated via

$$
\psi^{G,n,m+1}_{z,i,z',W} = \left(\lambda_i \frac{\widehat{\psi}^{p,n}_{z'}(\psi^{G,n,m}_{z,z',W}) + \theta z' + \overline{B}\widehat{\psi}^{q,n}_{z'}(\psi^{G,n,m}_{z,z',W}) - \overline{B}}{\psi^{p,n}_{z,W}} + \frac{(1-\lambda_i)}{\psi^{q,n}_{z,W}}\right) \psi^{y,n,m+1}_{i,z,W}.
$$

6 If $\max_{W\in\mathbb{W}} \max_z \max_{z'} \max_i |\psi^{y,n,m+1}_{z,i,W} - \psi^{y,n,m}_{z,i,W}|$ is below an acceptable tolerance level, then we proceed to step [7]. Otherwise we return to [4] with the updated savings functions and aggregate law of motion for wealth, but now indexed by step $m+1$. We now generate new Chebyshev coefficients $\alpha^{y,n,m+1}_{z,i}$ by solving the system $\widehat{\psi}^{y,n,m+1}_{z,i}(W) = \psi^{y,n,m+1}_{z,i,W}$ for each $W \in \mathbb{W}$.

7 For each point in the grid $\mathbb{W}$ and each value z, we check the market clearing conditions. If:

$$
\max_{W\in\mathbb{W}} \max_z |\sum_{i=1}^{I-1} \frac{\psi^{y,n,m+1}_{z,i,W}\lambda_i}{\psi^{p,n}_{z,W}} - 1| + |\sum_{i=1}^{I-1} \frac{\psi^{y,n,m+1}_{z,i,W}(1-\lambda_i)}{\psi^{q,n}_{z,W}} - \overline{B}|
$$

is below an acceptable tolerance level we stop. Otherwise, we update our guess of prices $\psi^{p,n+1}_{z,W} = \sum_{i=1}^{I-1} \lambda_i \psi^{y,n,m+1}_{z,i,W}$ and $\psi^{q,n+1}_{z,W} = \sum_{i=1}^{I-1}(1-\lambda_i)\psi^{y,n,m+1}_{z,i,W}/\overline{B}$ and return to step [3]. We now generate new Chebyshev coefficients $(\alpha^{p,n+1}_{z,i}, \alpha^{q,n+1}_{z,i}$ by solving $\widehat{\psi}^{p,n+1}_{z}(W) = \psi^{p,n+1}_{z,W}$ and $\widehat{\psi}^{q,n+1}_{z}(W) = \psi^{q,n+1}_{z,W}$ for each value of z and each $W \in \mathbb{W}$.

B.3 Numerical Accuracy

The analytical results available for the endogenous portfolio economy, for the case when $\gamma = 1$ and the age profile of earnings does not vary with z, provide us with a useful test case to assess the numerical accuracy of our computational results. We now compare our numerical results with the theoretical prediction from Proposition 1 in the main text, item by item. We make the following observations from our simulations:

1 The distribution of wealth shares is constant along the simulation. This is shown for the computed model in Table A-1.

Table A-1: WEALTH SHARES WITH $\gamma = 1$

	30-39	40-49	50-59	60-69	70-79
Expansion	6.1%	14.2%	25.3%	32.2%	22.3%
Recession	6.1%	14.2%	25.3%	32.2%	22.3%

2 Aggregate wealth is proportional to the aggregate shock. Specifically:

$$p(z, \bar{A}) + q(z, \bar{A})\bar{B} = 0.5501z.$$

3 The theoretical expressions for stock and bond prices hold with a maximal error of 0.002% in our simulation.

4 According to the proposition, with $\gamma = 1$ portfolio shares λ_i are age invariant and proportional to $\frac{p(z,\bar{A})}{z}$. The shares for each generation in the boom and recession states are shown to be age invariant in Table (A-2). The maximal deviation from the theoretical value is 0.001%.

Table A-2: PORTFOLIO SHARES WITH $\gamma = 1$

	20-29	30-39	40-49	50-59	60-69
Boom	0.9178	0.9178	0.9178	0.9178	0.9178
Recession	0.9178	0.9178	0.9178	0.9178	0.9178

5 Consumption profiles normalized by total output z (that is, consumption shares) are displayed in the first two columns of Table A-3. Note that they are independent of z, as the proposition indicates. They are also equal to the theoretical consumption shares characterized by Proposition 1 and displayed in the last column of Table A-3.

Table A-3: CONSUMPTION AS FRACTION OF OUTPUT

Age Group	Expansion	Recession	Theory
20-29	5.33%	5.33%	5.33%
30-39	8.92%	8.92%	8.92%
40-49	12.51%	12.51%	12.51%
50-59	20.55%	20.55%	20.55%
60-69	27.96%	27.96%	27.96%
70+	24.74%	24.74%	24.74%

6 The equity premium in theory is 0.6642% and the simulated equity premium is 0.6634%.

C Asset Prices in the Representative Agent Economy

Suppose the representative agent invests an exogenous fraction λ of savings in stocks and fraction $1-\lambda$ in bonds. Let $c(z,a)$ and $y(z,a)$ denote optimal consumption and savings as functions of the aggregate shock z and individual start-of-period wealth a, and let $p(z)$ and $q(z)$ be the equilibrium prices for stocks and bonds. Note that there is no need to keep track of aggregate wealth as a state: by assumption, the supply of capital is constant and equal to one. Thus, prices can only depend on z.

The dynamic programming problem for a household is

$$v(z,a) = \max_{c\geq 0,y} \left\{ u(c) + \beta \sum_{z'\in Z} \Gamma_{z,z'} \; v\left(z',a'\right)\right\}$$

subject to

$$c + y = (1-\theta)z + \left(p(z) + d(z) + B\right) a$$

and the law of motion

$$a' \left[p(z') + d(z') + B\right] = \left(\frac{\lambda \; [p(z') + d(z')]}{p(z)} + \frac{(1-\lambda)}{q(z)} \right) y.$$

The solution to this problem yields decision rules $c(z,a)$, $y(z,a)$ and $a'(z', y(z,a))$ is the associated value for next period wealth.

Given the preferences and technology described above, the market clearing conditions are simply

$$\begin{aligned}\lambda y(z,1) &= p(z),\\ (1-\lambda)y(z,1) &= q(z)B,\\ c(z,1) &= z.\end{aligned}$$

The individual and aggregate consistency condition is

$$a'(z', y(z,1)) = 1.$$

Now suppose the process for z is a two-state Markov chain. There are just two equity prices to solve for: $p(z) \in \{p(z_l), p(z_h)\}$. The two market clearing conditions for stocks and bonds imply a parametric relationship between $q(z)$ and $p(z)$:

$$q(z) = \frac{p(z)(1-\lambda)}{\lambda B}.$$

Thus, stock and bond prices must be equally sensitive to aggregate shocks. The realized gross real return to saving is given by

$$\lambda\frac{[p(z') + d(z')]}{p(z)} + (1-\lambda)\frac{1}{q(z)} = \frac{p(z') + \lambda\theta z'}{p(z)},$$

where the second equality follows from substituting in $d(z') = \theta z' + q(z')B - B$ and the expression for $q(z)$, as a function of $p(z)$. Thus, the equilibrium equity prices are defined by the solutions to the two intertemporal first-order conditions:

$$\begin{aligned}p(z_h)u'(c(z_h,1)) &= \beta \sum_{z'\in Z} \Gamma_{z_h,z'}\left[u'(c(z',1))\left[\lambda\theta z' + p(z')\right]\right],\\ p(z_l)u'(c(z_l,1)) &= \beta \sum_{z'\in Z} \Gamma_{z_l,z'}\left[u(c(z',1))\left[\lambda\theta z' + p(z')\right]\right],\end{aligned}$$

which, using the market clearing condition for consumption and the CRRA preference specification, can be written as

$$\begin{aligned}p(z_h)z_h^{-\gamma} &= \beta\Gamma_h\; z_h^{-\gamma}\left[\lambda\theta z_h + p(z_h)\right] + \beta(1-\Gamma_h)z_l^{-\gamma}\left[\lambda\theta z_l + p(z_l)\right],\\ p(z_l)z_l^{-\gamma} &= \beta\Gamma_l\; z_l^{-\gamma}\left[\lambda\theta z_l + p(z_l)\right] + \beta(1-\Gamma_l)z_h^{-\gamma}\left[\lambda\theta z_h + p(z_h)\right],\end{aligned}$$

where $\Gamma_h = \Gamma_{z_h,z_h}$ and $\Gamma_l = \Gamma_{z_l,z_l}$. From the second pricing equation,

$$p(z_h) = \frac{\beta\Gamma_h\lambda\theta z_h + \beta(1-\Gamma_h)\frac{z_l^{-\gamma}}{z_h^{-\gamma}}\left(\lambda\theta z_l + p(z_l)\right)}{(1-\beta\Gamma_h)}.$$

Substituting this into the first pricing equation,

$$p(z_l) = \frac{\beta\Gamma_l\ z_l^{-\gamma}\lambda\theta z_l + \beta(1-\Gamma_l)z_h^{-\gamma}\left(\lambda\theta z_h + \frac{\beta\Gamma_h\lambda\theta z_h + \beta(1-\Gamma_h)\frac{z_l^{-\gamma}}{z_h^{-\gamma}}\lambda\theta z_l}{(1-\beta\Gamma_h)}\right)}{z_l^{-\gamma}\left(\frac{(1-\beta)(1+\beta(1-\Gamma_h-\Gamma_l))}{(1-\beta\Gamma_h)}\right)}.$$

Since the expression for $p(z_h)$ is symmetric, we can take the ratio to express the ratio of prices across states as a function of fundamentals:

$$\frac{p(z_l)}{p(z_h)} = \frac{z_l}{z_h}\left(\frac{(1-\Gamma_l)\,z_h^{1-\gamma}z_l^{\gamma-1} + (\beta+\Gamma_l-\beta\Gamma_h-\beta\Gamma_l)}{(1-\Gamma_h)\,z_l^{1-\gamma}z_h^{\gamma-1} + (\beta+\Gamma_h-\beta\Gamma_h-\beta\Gamma_l)}\right).$$

Note that λ and θ have dropped out here: the *ratio* of stock prices across states does not depend on either λ or θ, though the *levels* of prices do. If aggregate shocks are *iid*, then $1-\Gamma_l = \Gamma_h$ and the expression above simplifies to

$$\frac{p(z_l)}{p(z_h)} = \left(\frac{z_l}{z_h}\right)^{\gamma}.$$

It is straightforward to verify that the same result is obtained even without the *iid* assumption in two special cases: $\gamma = 1$ or $\beta = 1$.

D Economy with Logarithmic Utility

If the economy is populated with households with logarithmic utility that have life cycle endowment profiles that do not depend on the aggregate shock, then we can solve for a recursive competitive equilibrium in closed form.

Proposition 3. *Assume (i) the period utility function is logarithmic ($\gamma = 1$), and (ii) relative earnings across age groups are independent of the aggregate state, $\varepsilon_i(z) = \varepsilon_i\ \forall z$.Then there exists a recursive competitive equilibrium in the economy with endogenous portfolio choice with the following properties:*

1. *The distribution of wealth A is constant over time. Denote this distribution* $\overline{A} = (\overline{A}_1, \ldots, \overline{A}_I)$. *Then*

$$G_{i+1}(z, \overline{A}, z') = a_i'(z, \overline{A}, \overline{A}_i, z') = \overline{A}_{i+1} \qquad \forall z, z', \ \forall i = 1, ..., I-1.$$

2. *Aggregate wealth is proportional to the aggregate shock:*

$$p(z, \overline{A}) + q(z, \overline{A})B = \Psi z \qquad \forall z,$$

where Ψ *is a constant that does not depend on the value for* B.

3. *Stock and bond prices are given by*

$$\begin{aligned} p(z, \overline{A}) &= p(z) = \Psi z - B\frac{z}{R}\sum_{z' \in Z} \Gamma_{z,z'}\frac{1}{z'} \\ q(z, \overline{A}) &= q(z) = \frac{z}{R}\sum_{z' \in Z} \Gamma_{z,z'}\frac{1}{z'} \qquad \forall z, \end{aligned}$$

where $R = (\Psi + \theta)/\Psi$ *is the nonstochastic steady state gross interest rate.*

4. *Asset portfolios are identical across age groups:*

$$\lambda_i(z, \overline{A}, \overline{A}_i) = \lambda(z) = \frac{p(z)}{\Psi z} \qquad \forall z, \ \forall i = 1, ..., I-1.$$

5. *Consumption and savings at each age are proportional to the aggregate shock:*

$$\begin{aligned} c_i(z, \overline{A}, \overline{A}_i) &= \left[(1-\theta)\varepsilon_i + \theta\overline{A}_i + \left(\overline{A}_i - \overline{A}_{i+1}\right)\Psi\right] z, \\ y_i(z, \overline{A}, \overline{A}_i) &= \overline{A}_{i+1}\Psi z \qquad \forall z, \ \forall i = 1, ..., I-1. \end{aligned}$$

6. *The equity premium is given by*

$$\sum_z \Pi_z \left\{ \sum_{z'} \Gamma_{z,z'} \frac{[p(z') + d(z')]}{p(z)} - \frac{1}{q(z)} \right\} = R \sum_z \frac{\Pi_z}{z} \left\{ \frac{\sum_{z' \in Z} \Gamma_{z,z'} z' - \left(\sum_{z' \in Z} \Gamma_{z,z'} \frac{1}{z'} \right)^{-1}}{1 - \frac{B}{R\Psi} \sum_{z' \in Z} \Gamma_{z,z'} \frac{1}{z'}} \right\}$$

where Π_z *denotes the unconditional probability distribution over* z.

Corollary 4. *If z is iid over time, then stock and bond prices are proportional to the aggregate shock and the average equity premium is given by* $R\left(\sum_z \frac{\Pi_z}{z}\sum_z \Pi_z z - 1\right) / \left(1 - \frac{B}{R\Psi}\sum_z \frac{\Pi_z}{z}\right)$.

Corollary 5. *In the limit as $\Gamma_{z,z} \to 1\ \forall z$ (perfectly persistent shocks), $q(z) \to R^{-1}$ and $p(z) \to \Psi z - BR^{-1}$.*

We will verify that the conjectured expressions for prices and allocations satisfy households' budget constraints, households' intertemporal first-order conditions, and all the market clearing conditions.

1. Market Clearing Recall that $\sum_{i=1}^{I} \varepsilon_i = 1$, $\overline{A}_1 = 0$ and $\sum_{i=1}^{I} \overline{A}_i = 1$. It is then straightforward to verify that the expressions in Proposition 1 for $\lambda_i(z)$, $c_i(z, \overline{A}, \overline{A}_i)$, and $y_i(z, \overline{A}, \overline{A}_i)$ satisfy the market clearing conditions for goods, stocks, and bonds.

2. Budget Constraints Given identical portfolios across age groups, all households earn the return to saving. Substituting in the candidate expressions for prices (Property 3) and portfolio shares (Property 4), the gross return to saving conditional on productivity being z_{-1} in the previous period and z in the current period is

$$\begin{aligned}\frac{\lambda(z_{-1})\left[p(z)+d(z)\right]}{p(z_{-1})} + \frac{1-\lambda(z_{-1})}{q(z_{-1})} &= \frac{p(z)+d(z)}{z_{-1}\Psi} + \frac{z_{-1}\Psi - p(z_{-1})}{z_{-1}\Psi q(z_{-1})} \\ &= \frac{p(z)+d(z)}{z_{-1}\Psi} + \frac{B}{z_{-1}\Psi} \\ &= \frac{z\left(\Psi+\theta\right)}{z_{-1}\Psi}.\end{aligned}$$

Given this expression for returns, consumption for a household of age i is

$$c_i(z, \overline{A}, \overline{A}_i) = (1-\theta)\varepsilon_i z + y_{i-1}(z_{-1}, \overline{A}, \overline{A}_i)\frac{z\left(\Psi+\theta\right)}{z_{-1}\Psi} - y_i(z, \overline{A}, \overline{A}_i).$$

Substituting in the candidate expression for $y_i(z, \overline{A}, \overline{A}_i)$ gives

$$c_i(z, \overline{A}, \overline{A}_i) = z\left[(1-\theta)\varepsilon_i + \theta\overline{A}_i + \left(\overline{A}_i - \overline{A}_{i+1}\right)\Psi\right],$$

which is the conjectured expression for equilibrium consumption (Property 5). Thus, the conjectured allocations satisfy households' budget constraints.

3. Optimal Savings and Portfolio Choices It remains to verify that agents' intertemporal first-order conditions with respect to stocks and bonds are satisfied. For bonds we have

$$\frac{q(z)}{c_i(z, \overline{A}, \overline{A}_i)} = \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'} \frac{1}{c_{i+1}(z', \overline{A}, \overline{A}_{i+1})} \qquad \forall i = 1, ..., I-1.$$

Substituting in the expression for consumption,

$$q(z) = \kappa_{i+1}(\overline{A}) \sum_{z' \in Z} \Gamma_{z,z'} \frac{z}{z'} \qquad \forall i = 1, ..., I-1, \tag{A-15}$$

where

$$\kappa_{i+1}(\overline{A}) = \beta_{i+1} \frac{(1-\theta)\varepsilon_i + \theta \overline{A}_i + \left(\overline{A}_i - \overline{A}_{i+1}\right)\Psi}{(1-\theta)\varepsilon_{i+1} + \theta \overline{A}_{i+1} + \left(\overline{A}_{i+1} - \overline{A}_{i+2}\right)\Psi}.$$

For stocks, we have

$$\begin{aligned} p(z) &= \kappa_{i+1}(\overline{A}) \sum_{z' \in Z} \Gamma_{z,z'} \frac{z}{z'} \left(p(z') + \theta z' + q(z')B - B\right) \qquad \text{(A-16)} \\ &= \kappa_{i+1}(\overline{A}) \sum_{z' \in Z} \Gamma_{z,z'} z \left(\Psi + \theta - \frac{B}{z'}\right) \qquad \forall i = 1, ..., I-1. \end{aligned}$$

Adding the two first-order conditions for stocks and bonds gives

$$\begin{aligned} p(z) + q(z)B &= z\Psi = \kappa_{i+1}(\overline{A}) \sum_{z' \in Z} \Gamma_{z,z'} \left[z \left(\Psi + \theta - \frac{B}{z'}\right) + \frac{z}{z'}B\right] \qquad \text{(A-17)} \\ &= \kappa_{i+1}(\overline{A}) z \left(\Psi + \theta\right) \qquad \forall i = 1, ..., I-1. \end{aligned}$$

This equation is satisfied as long as

$$\kappa_{i+1}(\overline{A}) = \frac{\Psi}{\Psi + \theta} = \frac{1}{R} \qquad \forall i = 1, ..., I-1$$

Given this expression for $\kappa_{i+1}(\overline{A})$, it is immediate that the expressions for asset prices (Property 3) satisfy the households' first-order conditions for stocks and bonds (equations A-15 and A-16):

$$
\begin{aligned}
p(z,\overline{A}) &= p(z) = z\Psi - B\frac{z}{R}\sum_{z'\in Z}\Gamma_{z,z'}\frac{1}{z'},\\
q(z,\overline{A}) &= q(z) = \frac{z}{R}\sum_{z'\in Z}\Gamma_{z,z'}\frac{1}{z'} \qquad \forall z.
\end{aligned}
$$

4. Equity Premium We can derive a near-closed-form expression for the equity premium (up to the endogenous value for wealth Ψ). Let Π_z denote the unconditional probability of aggregate productivity being z. The average equity premium is defined as

$$
\begin{aligned}
& \sum_z \Pi_z \left\{ \sum_{z'} \Gamma_{z,z'} \left[\frac{p(z') + d(z')}{p(z)} \right] - \frac{1}{q(z)} \right\} \\
= & \sum_z \Pi_z \left\{ \sum_{z'} \Gamma_{z,z'} \left[\frac{p(z') + \theta z' + q(z')B - B}{p(z)} \right] - \frac{1}{q(z)} \right\} \\
= & \sum_z \frac{\Pi_z}{p(z)} \left\{ \sum_{z'} \Gamma_{z,z'} \left[z' \left(\Psi + \theta \right) \right] - B - \frac{\sum_{z'\in Z} \Gamma_{z,z'} z \left(\Psi + \theta - \frac{B}{z'} \right)}{\sum_{z'\in Z} \Gamma_{z,z'} \frac{z}{z'}} \right\} \\
= & \left(\Psi + \theta \right) \sum_z \frac{\Pi_z}{p(z)} \left\{ \sum_{z'} \Gamma_{z,z'} z' - \frac{1}{\sum_{z'\in Z} \Gamma_{z,z'} \frac{1}{z'}} \right\} \\
= & R \sum_z \frac{\Pi_z}{z} \left\{ \frac{\left(\sum\limits_{z'\in Z} \Gamma_{z,z'} z' - \left(\sum\limits_{z'\in Z} \Gamma_{z,z'} \frac{1}{z'} \right)^{-1} \right)}{\left(1 - \frac{B}{R\Psi} \sum\limits_{z'\in Z} \Gamma_{z,z'} \frac{1}{z'} \right)} \right\}
\end{aligned}
$$

, where $R = \frac{(\Psi+\theta)}{\Psi}$. If z is *iid* so that $\Gamma_{z,z'}$ is equal to the unconditional probability $\Pi_{z'}$, this simplifies to

$$
\sum_z \Pi_z \left\{ \sum_{z'} \Pi_{z'} \left[\frac{p(z') + d(z')}{p(z)} \right] - \frac{1}{q(z)} \right\} = R \frac{\left(\sum\limits_z \frac{\Pi_z}{z} \sum\limits_z \Pi_z z - 1 \right)}{\left(1 - \frac{B}{R\Psi} \sum\limits_z \frac{\Pi_z}{z} \right)}.
$$

5. Solving for Ψ and $\overline{A}$ Equations (A-17) are the first-order conditions for pricing claims to capital for a nonstochastic life-cycle economy, in which the constant asset price is Ψ and the constant asset income is θ. The $I-1$ equations (A-17) combined with $\overline{A}_1 = 0$ and

the market clearing condition $\sum_{i=1}^{I} \overline{A}_i = 1$ can be used to solve numerically for $\{\overline{A}_i\}_{i=1}^{I}$ and Ψ. This system of equations is the one used to calibrate the nonstochastic version of our model economy. There we set θ to replicate a target interest rate $R = (\Psi + \theta)/\Psi$, and we set the life-cycle profile $\{\beta_i\}_{i=2}^{I}$ to replicate the empirical distribution for wealth by age, which determines both the aggregate start-of-period wealth $(\Psi + \theta)$ and its age distribution $\{\overline{A}_i\}_{i=1}^{I}$.

E Asset Prices in the Two-Period Overlapping-Generations Economy

In this appendix we study the simplest OLG framework in which households live for only two periods: $I = 2$. We use this example to discuss how the curvature parameter γ affects the elasticity ξ of price changes to output changes in OLG economies. To make that discussion most transparent, we focus on an economy with only the risky stock ($B = 0$ and $\lambda_i \equiv 0$) and assume that households only earn labor income in the first period of life: $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$. Since young households start with zero assets, all wealth is therefore held by old agents. As a consequence, the wealth distribution is degenerate (and time invariant) in this economy. As in the representative agent model, the only state variable is the exogenous shock $z \in \{z_l, z_h\}$.

Consumption of young and old households is given by

$$\begin{aligned} c_1(z) &= (1-\theta)z - p(z) \\ c_2(z) &= \theta z + p(z), \end{aligned}$$

and the stock market price is determined by the intertemporal Euler equation

$$p(z)\ \left[(1-\theta)z - p(z)\right]^{-\gamma} = \beta \sum_{z' \in \{z_L, z_h\}} \Gamma_{z,z'}\ \left[\theta z' + p(z')\right]^{-\gamma}\ \left[\theta z' + p(z')\right]. \tag{A-18}$$

No closed-form solution is available for the functional equation $p(z)$ that solves equation $(A-18)$ outside of the special cases $\gamma = 0$ and $\gamma = 1$. However, taking a first order approximation of the Euler equation around the point $z_l/z_h = 1$ we can show

Proposition 6. *To a first order approximation*

$$\begin{aligned}\xi^{2p} &\approx \frac{\gamma(1-\theta)}{1-\theta\frac{(R-\gamma)}{(R-1)}} \\ &= \xi^{RA}\times\frac{1-\theta}{1-\theta\frac{(R-\gamma)}{(R-1)}},\end{aligned}$$

where $R=\frac{\theta+p}{p}>1$ is the steady state gross return on the stock.[54]

We prove this proposition in section below. Note first that for $\gamma=1$, this formula is exact (as shown in the previous section) and delivers $\xi^{2p}=\xi^{RA}=1$: prices fall by exactly as much as output in a downturn. Second, ξ^{2p} is increasing in γ, and thus for $\gamma>1$ we have $\xi^{2p}>1$. Third, $\xi^{2p}<\xi^{RA}$. Thus, as long as the intertemporal elasticity of substitution $1/\gamma$ is smaller than one, asset prices fall by more than output in a recession, but by less than in the corresponding representative agent economy with infinitely lived households.

The reason is as follows. Consumption of the current old generation must decline in the recession since the price of the asset, the only source for old-age consumption, is lower in the bad than in the good aggregate state of the world. Moreover, for $\gamma>1$, consumption of the old is more sensitive to aggregate shocks than consumption of the young:

$$\frac{c_1(z_h)}{c_1(z_l)}<\frac{z_h}{z_l}<\frac{c_2(z_h)}{c_2(z_l)}.$$

The second inequality reflects the fact that $c_2(z_h)/c_2(z_l)=p_h/p_l>z_h/z_l$ (since $\xi^{2p}>1$), while the first inequality follows from market clearing: $(c_1(z_h)+c_2(z_h))\,/\,(c_1(z_l)+c_2(z_l))=z_h/z_l$. The fact that aggregate risk is disproportionately borne by the old explains why stock prices are less volatile in this economy than in the analogous representative agent economy. Recall that stocks are effectively priced by younger agents, because the supply of stocks by the old is inelastic at any positive price. Because the old bear a disproportionate share of aggregate risk, the young's consumption fluctuates less than output. Thus, smaller price changes (relative to the representative agent economy) are required to induce them to purchase the aggregate supply of equity at each date.

One might wonder whether it is possible that $c_1(z_h)<c_1(z_l)$, so that newborn households would potentially prefer to enter the economy during a recession rather than during a boom. The

[54] For θ such that $R=\beta^{-1}$, the expression simplifies to $\xi^{2p}\approx\frac{\gamma(\beta+1)}{\gamma\beta+1}$.

answer turns out to be no: while stock prices fall by more than output in the event of a recession, they never fall by enough to compensate the young for their decline in labor earnings. The logic for this result is straightforward. In a two-period OLG economy, stock prices are defined by the inter-temporal first-order condition for young households (equation A-18). With *iid* shocks, the right-hand side of this condition is independent of the current value for z. Taking the ratio of the two pricing equations across states, the ratio of stock prices across states is given by

$$\frac{p_h}{p_l} = \left(\frac{c_1(z_h)}{c_1(z_l)}\right)^{\gamma}.$$

The advantage to the young from entering the economy during a recession is that they buy stocks cheaply, $p_h/p_l > 1$. But the optimality restriction above then implies that $c_1(z_h)/c_1(z_l) > 1$, so the young must suffer low consumption if they enter during a recession. Intuitively, low prices are needed to induce the young to buy stocks when the marginal utility of current consumption is high. But a high marginal utility of consumption requires low consumption for these households.

This example reveals that for the young to potentially gain from a recession, we need people to live for at least three periods, while the previous example with logarithmic preferences indicates that we also require $\gamma > 1$. That is why, in the main text, we focus on the 3 period version of the model in which both desired results can emerge simultaneously.

E.1 Proof of Proposition 6

Let $\widetilde{p} = \frac{p(z_h)}{p(z_l)}$, $\widetilde{z} = \frac{z_h}{z_l}$, and $\widetilde{x} = \frac{z_l}{p_l}$. In terms of these variables, the intertemporal first-order conditions, conditional on the current state being z_l and z_h are, respectively,

$$\begin{aligned} ((1-\theta)\widetilde{x}-1)^{-\gamma} &= \beta\Gamma_{z_l,z_l}\ (\theta\widetilde{x}+1)^{1-\gamma} + \beta\Gamma_{z_l,z_h}\ (\theta\widetilde{z}\widetilde{x}+\widetilde{p})^{1-\gamma} \\ \widetilde{p}\,((1-\theta)\widetilde{z}\widetilde{x}-\widetilde{p})^{-\gamma} &= \beta\Gamma_{z_h,z_l}\ (\theta\widetilde{x}+1)^{1-\gamma} + \beta\Gamma_{z_h,z_h}\ (\theta\widetilde{z}\widetilde{x}+\widetilde{p})^{1-\gamma}. \end{aligned}$$

Our goal is to solve for $\widetilde{p}$ as a function of $\widetilde{z}$. However, except for the special case $\gamma = 1$, this system of equations cannot be solved in closed form. So instead we will linearize these equations and look for an approximate solution for relative prices as a linear function of relative productivity. We proceed as follows:

1. Take first-order Taylor-series approximations to these two first-order conditions around the nonstochastic steady state values for $\widetilde{p}$, $\widetilde{z}$, and $\widetilde{x}$, which we denote P, Z, and X (where

$Z = P = 1$). This gives a system of two equations in three first-order terms $(\widetilde{x} - X)$, $(\widetilde{z} - Z)$, and $(\widetilde{p} - P)$:

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \begin{pmatrix} (\widetilde{x} - X) \\ (\widetilde{z} - Z) \\ (\widetilde{p} - P) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{A-19}$$

where

$$\begin{aligned}
A_{11} &= -\gamma\left((1-\theta)X - 1\right)^{-\gamma-1}(1-\theta) - \\
&\qquad \left((1-\gamma)\,\beta\Gamma_{z_l,z_l}\,(\theta X + 1)^{-\gamma}\,\theta + (1-\gamma)\,\beta\Gamma_{z_l,z_h}\,(\theta X + 1)^{-\gamma}\,\theta\right) \\
A_{12} &= -(1-\gamma)\,\beta\Gamma_{z_l,z_h}\,(\theta X + 1)^{-\gamma}\,\theta X \\
A_{13} &= -(1-\gamma)\,\beta\Gamma_{z_l,z_h}\,(\theta X + 1)^{-\gamma} \\
A_{21} &= -\gamma\left((1-\theta)X - 1\right)^{-\gamma-1}(1-\theta) - \\
&\qquad \left((1-\gamma)\,\beta\Gamma_{z_h,z_l}\,(\theta X + 1)^{-\gamma}\,\theta + (1-\gamma)\,\beta\Gamma_{z_h,z_h}\,(\theta X + 1)^{-\gamma}\,\theta\right) \\
A_{22} &= -\gamma\left((1-\theta)X - 1\right)^{-\gamma-1}(1-\theta)X - (1-\gamma)\,\beta\Gamma_{z_h,z_h}\,(\theta X + 1)^{-\gamma}\,\theta X \\
A_{23} &= \gamma\left((1-\theta)X - 1\right)^{-\gamma-1} + \left((1-\theta)X - P\right)^{-\gamma} - (1-\gamma)\,\beta\Gamma_{z_H,z_H}\,(\theta X + 1)^{-\gamma}.
\end{aligned}$$

2. Use the first equation in A-19 to solve for $(\widetilde{x} - X)$ as a linear function of $(\widetilde{z} - Z)$ and $(\widetilde{p} - P)$:

$$(\widetilde{x} - X) = -\frac{A_{12}}{A_{11}}(\widetilde{z} - Z) - \frac{A_{13}}{A_{11}}(\widetilde{p} - P).$$

Then substitute this solution into the second equation in (A-19), and solve for $(\widetilde{p} - P)$ as a function of $(\widetilde{z} - Z)$:

$$\begin{aligned}
(\widetilde{p} - P) &= -\frac{A_{21}}{A_{23}}(\widetilde{x} - X) - \frac{A_{22}}{A_{23}}(\widetilde{z} - Z) \\
&= -\frac{A_{21}}{A_{23}}\left(-\frac{A_{12}}{A_{11}}(\widetilde{z} - Z) - \frac{A_{13}}{A_{11}}(\widetilde{p} - P)\right) - \frac{A_{22}}{A_{23}}(\widetilde{z} - Z).
\end{aligned}$$

Thus,

$$\xi^{2p} \approx \frac{\widetilde{p} - P}{\widetilde{z} - Z} = \frac{A_{21}A_{12} - A_{22}A_{11}}{A_{23}A_{11} - A_{21}A_{13}}.$$

3. Now assume productivity shocks are *iid*, so that $\Gamma_{z_l,z_h} = \Gamma_{z_h,z_h} = \Gamma_{z_h}$ and $\Gamma_{z_l,z_l} = \Gamma_{z_h,z_l} = 1 - \Gamma_{z_h}$. Under this *iid* assumption, $A_{11} = A_{21}$ and thus

$$\xi^{2p} \approx \frac{A_{21}A_{12} - A_{22}A_{21}}{A_{23}A_{11} - A_{21}A_{13}} = \frac{A_{12} - A_{22}}{A_{23} - A_{13}} = X\frac{\gamma(1-\theta)}{((X - X\theta - 1) + \gamma)}$$

Recall that X is the inverse of the steady state stock price, so we can equivalently write this elasticity in terms of the steady state gross interest rate R, where $R = \theta X + 1$:

$$\xi^{2p} \approx \frac{\gamma(1-\theta)}{1 - \theta\frac{(R-\gamma)}{(R-1)}}. \tag{A-20}$$

This is the expression given in the text. Note that for $\gamma = 1$, $\xi^{2p} = 1$.

4. We want to show that $1 < \xi^{2p} < \xi^{RA}$ for $\gamma > 1$. First, note that in any equilibrium, a positive stock price implies $R > 1$. Then

$$\frac{1}{\xi^{2p}} = \frac{1 - \theta\frac{(R-\gamma)}{(R-1)}}{\gamma(1-\theta)} = \frac{1}{\gamma}\left(1 + \frac{(\gamma-1)\frac{\theta}{R-1}}{(1-\theta)}\right) > \frac{1}{\gamma} = \frac{1}{\xi^{RA}}.$$

Thus, $\xi^{2p} < \xi^{RA}$.

Given that $\xi^{2p} = 1$ when $\gamma = 1$, showing that ξ^{2p} is strictly increasing in γ is sufficient to prove that $\xi^{2p} > 1$:

$$\frac{\partial}{\partial\gamma}\left(\frac{\gamma(1-\theta)}{1 - \theta\frac{(R-\gamma)}{(R-1)}}\right) = (\theta - 1)(R-1)\frac{R\theta - R + 1}{(R - R\theta + \theta\gamma - 1)^2}$$

It follows that ξ^{2p} is strictly increasing in γ if and only if $R > \frac{1}{1-\theta}$. But in any equilibrium, positive consumption for the young requires exactly this condition:

$$(1-\theta) - \frac{\theta}{R-1} > 0 \Leftrightarrow R > \frac{1}{1-\theta}.$$

We conclude that $\xi^{2p} > 1$.

5. In the special case in which θ is such that $R = \frac{1}{\beta}$, the expression for ξ^{2p} simplifies further. The steady state value for R is an endogenous variable and has to satisfy the steady state version

of the intertemporal first-order condition, where the steady state stock price is $\theta/(R-1)$:

$$\frac{\theta}{R-1}\left((1-\theta)-\frac{\theta}{R-1}\right)^{-\gamma}=\beta\left(\theta+\frac{\theta}{R-1}\right)^{-\gamma}\left(\theta+\frac{\theta}{R-1}\right). \tag{A-21}$$

When $\beta=\frac{1}{R}$, equation (A-21) can be solved in closed form to give $R=\frac{1}{1-2\theta}$. Thus, we have $R=\frac{1}{\beta}=\frac{1}{1-2\theta}$, which implies $\theta=\frac{1}{2}(1-\beta)$. Substituting $R=\frac{1}{\beta}$ and $\theta=\frac{1}{2}(1-\beta)$ into equation (A-20) gives

$$\xi^{2p}\approx\frac{\gamma(1-\theta)}{1-\theta\frac{(R-\gamma)}{(R-1)}}=\frac{\gamma(\beta+1)}{\gamma\beta+1},$$

F Derivations for the 3-Period Economy

In this appendix we provide details of the analysis of the three period model, first for the case where aggregate productivity z follows as stationary Markov process in levels (studied in section 4 of the paper), and then for the case of stochastic growth rates of z (as investigated in section 7.1).

F.1 Details of the 3-Period Economy with Stationary Productivity

As discussed in the main text, the two functional equations fully characterizing the recursive equilibrium in the 3-period OLG economy are given by the intertemporal Euler equation

$$u'\left[(1-A)\left(p(z,A)+\theta z\right)-G(z,A)p(z,A)\right] \;=\; \beta\sum_{z'}\Gamma_{z,z'}\frac{[p(z',A')+\theta z']}{p(z,A)}u'\left[G(z,A)\left(p(z',A')+\theta z'\right)\right] \tag{A-22}$$

and the (rewritten) asset market clearing condition that the labor income of the young equals that cohort's purchases of shares in the risky asset:

$$[1-G(z,A)]\;p(z,A)=(1-\theta)z. \tag{A-23}$$

These two equations determine the equilibrium price $p(z,A)$ of the risky asset and the law of motion of the wealth distribution $A'=G(z,A)$. The young do not consume by assumption, and consumption of the middle-aged and old are given by

$$c_3(z,A) = A\left[p(z,A)+\theta z\right] \tag{A-24}$$

$$c_2(z,A) = (1-A)\left[p(z,S)+\theta z\right] - G(z,A)p(z,A) \tag{A-25}$$

and expected lifetime utility is therefore determined as

$$v_3(z,A) = u\left[c_3(z,A)\right] \tag{A-26}$$

$$v_2(z,A) = u\left[c_2(z,A)\right] + \beta \sum_{z'} \Gamma_{z,z'}\ u[c_3(z', G(z,A))] \tag{A-27}$$

$$v_1(z,A) = \beta \sum_{z'} \Gamma_{z,z'}\ v_2\left[z', G(z,A)\right]. \tag{A-28}$$

The welfare losses or gains from a great recession for the young are therefore derived from comparing $v_1(z_n, A)$ and $v_1(z_r, A)$, and the magnitude of the relative asset price decline is measured as the percentage decline in the price of the risky asset, relative to that of output:

$$\xi(A) = \frac{\log(p(z_r, A)/p(z_n, A))}{\log(z_r/z_n)}$$

F.2 Details of the 3-Period Economy with Stochastic Growth Rates

In the model with stochastic growth rate, assume that the growth rate (between today and next period) of aggregate productivity

$$g' = \frac{z'}{z}$$

follows a finite state Markov chain with state space $\mathcal{G}$ and transition matrix $\Gamma_{g,g'}$. Now the state space of the economy consists of the current growth rate g and again the wealth distribution, represented by the share of wealth (risky assets) of the old A coming into the period. To compute the model it is easier to work with variables that are deflated by current productivity z. Therefore define

$$\tilde{p}(g,A) = \frac{p(z,A)}{z}$$
$$\tilde{c}_m(g,A) = \frac{c_m(z,A)}{z}$$
$$\tilde{c}_o(g,A) = \frac{c_o(z,A)}{z}$$

As before, the Euler equation can now be written as

$$[(1-A)(p(z,A)+\theta z) - G(z,A)p(z,A)]^{-\gamma} = \beta \sum_{z'} \Gamma_{z,z'} \frac{[p(z',A')+\theta z']}{p(z,A)} [G(z,A)(p(z',A')+\theta z')]^{-\gamma}$$

and dividing by $z^{-\gamma}$ yields

$$\left[(1-A)\left(\frac{p(z,A)}{z}+\theta\right) - G(z,A)\frac{p(z,A)}{z}\right]^{-\gamma} = \beta \sum_{z'} \Gamma_{z,z'} \left(\frac{z'}{z}\right)^{1-\gamma} \frac{[p(z',A')/z'+\theta]}{p(z,A)/z} \left[G(z,A)\left(\frac{p(z',A')}{z'}+\theta\right)\right]^{-\gamma}$$

In terms of the deflated price, and now making the switch in the state variable from z to g, this yields the Euler equation

$$[(1-A)(\tilde{p}(g,A)+\theta) - G(g,A)\tilde{p}(g,A)]^{-\gamma} = \beta \sum_{g'} \Gamma_{g,g'} (g')^{1-\gamma} \frac{[\tilde{p}(g',A')+\theta]}{\tilde{p}(g,A)} [G(g,A)(\tilde{p}(g',A')+\theta)]^{-\gamma}$$

or

$$1 = \beta \sum_{g'} \Gamma_{g,g'} (g')^{1-\gamma} \frac{[\tilde{p}(g',A')+\theta]}{\tilde{p}(g,A)} \left[\frac{G(g,A)(\tilde{p}(g',A')+\theta)}{(1-A)(\tilde{p}(g,A)+\theta) - G(g,A)\tilde{p}(g,A)}\right]^{-\gamma}$$

The previous asset market clearing condition $(A-23)$ now reads as

$$[1-G(g,A)]\tilde{p}(g,A) = (1-\theta),$$

and thus again we have two functional equations which determine the equilibrium price function $\tilde{p}(g,A)$ and law of motion for wealth $G(g,A)$.

G Empirical Asset Price Movements and Long-Run Return Statistics

In the next table we summarize the evolution of asset prices, over the last decade, for the broad asset classes used in the empirical analysis in Section 2.

Next we document in greater detail the empirical asset return statistics for stocks and bonds, over 10 year time intervals. As described in the main text, the Shiller data provide annual real returns on stocks and bonds from 1871 to 2014. From these annual gross real returns we can construct 10 year returns, for an arbitrary 10 year interval, by taking the product of 10 contiguous one year gross real returns. If we insist on non-overlapping 10 year time intervals, there are 5 different ways to construct our sample of 14 observations, starting in 1871 and ending in 2010, starting in 1872 and ending in 2011 and so forth. The following table reports the range of the mean and standard deviation of stock and bond returns, as well as their correlation, across these five different samples. We also calculate the same statistics based on a sample of yearly (and thus overlapping) 10 year returns, where the first observation uses annual return data from 1871 to 1880, the second observation data from 1872 to 1881 and so forth.

Finally, once we have constructed samples of 10 year gross returns R, we compute means and standard deviations based on the log of gross real returns $\ln(R)$, as is common in the literature (see e.g. Campbell,) where we note that

$$\ln(R_{t,t+10}) = \sum_{\tau=t}^{t+9} \ln(R_{\tau,\tau+1}) = \sum_{\tau=t}^{t+9} r_{\tau,\tau+1}$$

where $r_{\tau,\tau+1} = \ln(R_{\tau,\tau+1})$ is the annual net real return. In order to make the mean returns easier to interpret we annualize it through the transformation $\exp\left(E\left[\ln R\right]\right)^{\frac{1}{10}}$ and express it in percentage terms. In the main text we report statistics based on the 1974-2014 sample (the middle columns in table A12. As discussed in the main text, the range of possible empirical return statistics we could have reported is quite narrow for average returns and bond return volatility, slightly wider for stock return volatility and appreciably larger for the correlation between 10 year real stock and bond returns (which is why we do not emphasize that statistic in our discussion of the model-based results).

Table A-4: Real Price Declines Relative to 2007:2 by Risky Asset Class

	Stocks: Wilshire 5000	Res. Real Estate: Case Shiller	Noncorp Bus.: Flow of Funds	Nonres. Property: Moody's	Net Worth	Trend Net Worth	Net Worth - Trend Net Worth
to 2008:1	-11.28	-8.07	-5.46	1.30	-7.77	1.50	-9.13
to 2008:2	-9.79	-9.79	-9.77	-2.06	-8.78	2.00	-10.57
to 2008:3	-17.72	-12.00	-11.42	-8.31	-12.64	2.51	-14.78
to 2008:4	-41.10	-16.48	-15.66	-15.47	-22.61	3.01	-24.87
to 2009:1	-47.66	-21.00	-20.18	-23.76	-27.59	3.53	-30.05
to 2009:2	-41.61	-20.62	-24.91	-31.86	-26.46	4.04	-29.32
to 2009:3	-34.45	-19.53	-28.23	-36.88	-24.37	4.56	-27.66
to 2009:4	-28.50	-21.16	-28.23	-39.10	-23.39	5.08	-27.09
to 2010:1	-25.96	-23.48	-28.49	-39.34	-23.72	5.60	-27.76
to 2010:2	-25.11	-22.35	-26.46	-38.05	-22.59	6.12	-27.05
to 2010:3	-28.17	-22.98	-25.76	-36.57	-23.63	6.65	-28.39
to 2010:4	-20.91	-25.63	-24.66	-35.33	-22.46	7.18	-27.65
to 2011:1	-14.54	-27.86	-23.71	-34.23	-21.38	7.71	-27.01
to 2011:2	-14.02	-27.16	-24.25	-33.40	-20.90	8.24	-26.92
to 2011:3	-20.95	-27.15	-23.80	-32.34	-22.86	8.78	-29.09
to 2011:4	-21.47	-29.66	-22.07	-31.36	-23.91	9.32	-30.39
to 2012:1	-13.73	-31.08	-21.68	-30.63	-22.12	9.86	-29.11
to 2012:2	-14.39	-28.20	-20.53	-29.22	-20.73	10.41	-28.21
to 2012:3	-11.89	-26.83	-18.04	-28.59	-18.95	10.96	-26.96
to 2012:4	-11.07	-27.35	-16.44	-27.21	-18.66	11.51	-27.05
to 2013:1	-4.83	-26.68	-14.94	-25.47	-16.16	12.06	-25.18
to 2013:2	0.70	-22.89	-12.29	-22.13	-12.21	12.62	-22.04
to 2013:3	4.81	-20.58	-9.52	-19.42	-9.38	13.18	-19.93
to 2013:4	10.35	-20.79	-7.54	-16.87	-7.42	13.74	-18.60
to 2014:1	14.34	-20.71	-6.95	-14.67	-5.98	14.30	-17.74
to 2014:2	17.19	-18.93	-5.80	-10.45	-3.94	14.87	-16.37
to 2014:3	20.98	-17.95	-4.61	-7.82	-2.04	15.44	-15.14
to 2014:4	22.52	-18.26	-2.57	-4.67	-1.30	16.01	-14.92
to 2015:1	26.34	-18.12	-0.62	-0.77	0.35	16.59	-13.92
to 2015:2	28.19	-16.21	-0.16	2.40	2.03	17.17	-12.92
to 2015:3	23.02	-14.98	0.56	4.77	1.26	17.75	-14.00
to 2015:4	23.47	-14.96	1.70	6.76	1.66	18.33	-14.09
avg. (2009:1--2013:4)							
	-18.17	-24.65	-21.09	-30.59	-20.44	8.56	-26.77

The Flow of Funds also reports price changes for directly held corporate equities: this series aligns closely with the Wilshire 5000 index. The Flow of Funds also reports a price series for residential real estate, based on the Loan Performance Index from First American Corelogic. This series closely tracks the Case-Shiller series. The house price series published by OFHEO (based on data from Fannie Mae and Freddie Mac) shows significantly smaller declines in house values.

Table A-5: Empirical Return Statistics

	Mean Return: $100\left[\exp\left(E\left[\ln R\right]\right)^{\frac{1}{10}}-1\right]$			Std.Dev.($\ln R$)		
	overlap $1871-2014$	non-overlap $1874-2014$	non-overlap [*min*, *max*]	overlap $1871-2014$	non-overlap $1874-2014$	non-overlap [*min*, *max*]
Equity	6.52	6.62	[6.43,6.62]	0.48	0.36	[0.35,0.55]
Bonds	2.36	2.29	[2.29,2.58]	0.31	0.30	[0.30,0.36]
Corr.	0.33	0.01	[0.01,0.52]			

Table A-6: Wealth-Based Welfare Losses (%)

	Portfolio & Earnings Model			
	Endogenous Port.		Exogenous Port.	
Age Group	Asym.	Sym.	Asym.	Sym.
20-29	-1.98	0.60	-3.90	-2.98
30-39	-11.20	-11.87	-6.30	-5.34
40-49	-15.79	-16.38	-6.83	-7.29
50-59	-22.83	-23.31	-20.39	-18.86
60-69	-25.90	-26.24	-35.77	-36.78
70+	-14.95	-15.08	-19.11	-20.90

H Wealth-Based Welfare Measures

I Welfare with Fixed Prices

In this appendix we provide the details of our calculations for the partial equilibrium thought experiments in section 6.5 of the main text.

I.1 No Asset Price Recession

In the first scenario, there is a recession at 0 but nothing happens to asset prices. So in the recession period, the wealth distribution is again A_{-1} and aggregate start of period wealth is

$$W_{-1} = p(z_H, A_{-1}) + \theta z_H + q(z_H, A_{-1})B$$

Thus, the age distribution of start of period wealth in the recession period is exactly what it would have been given no recession in the general equilibrium model.

Now, starting with their age-specific wealth, $A^i_{-1}W_{-1}$, agents at each age maximize expected lifetime utility looking forward, taking as given the stochastic process for z with the corresponding implications for their earnings. They believe that at each date and in each state, one period ahead gross returns will be R, where R is defined below.

We compare welfare conditional on a recession at date 0 to welfare without a recession at date 0. When there is no recession at date, households also have start of period wealth $A^i_{-1}W_{-1}$, but they have higher labor earnings (and a different probability distribution over future earnings). They take as given the same gross returns R looking forward.

The date 0 constraints for an agent of age i with (without) a recession are

$$\begin{aligned} c_i + y_i &= \varepsilon_i(z_R)(1-\theta)z_R + A^i_{-1}W_{-1} \\ c_i + y_i &= \varepsilon_i(z_R)(1-\theta)z_H + A^i_{-1}W_{-1} \end{aligned}$$

where y_i is savings. We will later assume savings are divided between $z'-$contingent Arrow securities, and that the cost of buying one of each of these securities (thereby delivering one unit of consumption tomorrow) is $1/R$.

I.2 Permanent Asset Price Recession

In the second scenario, the distribution of start of period wealth in the recession moves just as it does in the baseline general equilibrium model. Call this recession distribution, A_R, where A_R is the distribution of start of period wealth conditional on a long period of normal times prior to date 0, followed by a recession at date zero. Aggregate start of period wealth is

$$W_R = p(z_R, A_R) + \theta z_R + q(z_R, A_R)B$$

Thus, the age distribution of start of period wealth in the recession period is exactly what it would have been given a recession in the general equilibrium model.

Now, starting with their age-specific wealth, $A^i_R W_R$, agents at each age maximize expected lifetime utility looking forward, taking as given the stochastic process for z with the corresponding implications for their earnings. They believe that at each date and in each state, one period ahead returns will be R (as in the other scenario).

The date 0 constraints for an agent of age i with (without) a recession are

$$
\begin{aligned}
c_i + y_i &= \varepsilon_i(z_R)(1-\theta)z_R + A^i_R W_R \\
c_i + y_i &= \varepsilon_i(z_H)(1-\theta)z_H + A^i_{-1} W_{-1}
\end{aligned}
$$

I.3 Returns and Risk Sharing

One issue is that if there is no uncertainty about stock or bond returns, and equities offer a return premium, everyone will want to short bonds, if they are allowed to do so. Thus we need to either impose exogenous portfolios or assume that both assets must pay the same constant return from the recession period onward. In either case it seems reasonable to set the return to saving equal to the aggregate return to assets in the repeated normal state, i.e.,

$$R = \frac{p(z_H, A_{-1}) + d(z_H, A_{-1}) + B}{p(z_H, A_{-1}) + q(z_H, A_{-1})B}.$$

A second issue is that with constant returns, there is no way to pool aggregate risk. This introduces an asymmetry relative to the baseline model, which we probably don't want. Suppose we introduce Arrow securities, which pay one unit of consumption if a particular aggregate state is realized, and which are priced at actuarially fair rates, so the price of consumption in z' given z today is

$$q(z, z') = \frac{pr(z'|z)}{R}$$

Thus the price of one unit of consumption for sure is simply

$$\sum_{z'} q(z, z') = \frac{1}{R}$$

Given these assumptions, the household problem at age i and date 0 given initial productivity

z_0 and initial start of period wealth distribution $A^i W$ is

$$\max_{\{c_j(z^{0+j-i})\}} \left\{ u(c_i(z_0)) + \sum_{j=i+1}^{I} (\beta_i \times ... \times \beta_{j-1}) \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) u(c_j(z^{0+j-i})) \right\}$$

subject to

$$c_i(z_0) + \sum_{j=i+1}^{I} \frac{1}{R^{j-i}} \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) c_j(z^{0+j-i}) \leq LTI^i_{z_0}$$

$$LTI^i_{z_0} \equiv A^i W + \varepsilon_i(z_0)(1-\theta)z_0 + \sum_{j=i+1}^{I} \frac{1}{R^{j-i}} \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0)\varepsilon_i(z\left(z^{0+j-i}\right))(1-\theta)z\left(z^{0+j-i}\right)$$

where z^{0+j-i} is a possible history from date zero to date $0+j-i$, and $\pi_0(z^{0+j-i}|z_0)$ is the corresponding probability conditional on z_0 (which matters because shocks are not *iid*) and where $z\left(z^{0+j-i}\right)$ is just the last element of the sequence. Note that I have written the consumption prices straight into the budget constraint.

The first order conditions are

$$\begin{aligned} u'(c_i(z_0)) &= \lambda \\ (\beta_i \times ... \times \beta_{j-1})\, \pi_0(z^{0+j-i}|z_0) u'(c_j(z^{0+j-i})) &= \lambda \frac{1}{R^{j-i}} \pi_0(z^{0+j-i}|z_0) \end{aligned}$$

which imply

$$c_j(z^{0+j-i}) = \left[R^{j-i}\left(\beta_i \times ... \times \beta_{j-1}\right)\right]^{\frac{1}{\gamma}} c_i(z_0)$$

We also have

$$\begin{aligned} c_i(z_0) + \sum_{j=i+1}^{I} \frac{1}{R^{j-i}} \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) c_j(z^{0+j-i}) &= LTI^i_{z_0} \\ c_i(z_0) + \sum_{j=i+1}^{I} \frac{1}{R^{j-i}} \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) \left[R^{j-i}\left(\beta_i \times ... \times \beta_{j-1}\right)\right]^{\frac{1}{\gamma}} c_i(z_0) &= LTI^i_{z_0} \\ c_i(z_0) \left(1 + \sum_{j=i+1}^{I} \frac{1}{R^{j-i}} \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) \left[R^{j-i}\left(\beta_i \times ... \times \beta_{j-1}\right)\right]^{\frac{1}{\gamma}} \right) &= LTI^i_{z_0} \end{aligned}$$

or

$$c_i(z_0) = \chi^i_{z_0} LTI^i_{z_0}$$

where

$$\chi^i_{z_0} = \frac{1}{\left(1 + \sum_{j=i+1}^{I} \frac{1}{R^{j-i}} \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) \left[R^{j-i} \left(\beta_i \times ... \times \beta_{j-1}\right)\right]^{\frac{1}{\gamma}}\right)}$$

But note here that in fact nothing in $\chi^i_{z_0}$ depends on the history, and the probabilities add to one, so we can write

$$\chi^i_{z_0} = \chi^i = \frac{1}{1 + \sum_{j=i+1}^{I} \left(R^{j-i}\right)^{\frac{1}{\gamma}-1} \left(\beta_i \times ... \times \beta_{j-1}\right)^{\frac{1}{\gamma}}}$$

I.4 Lifetime Utility and Welfare Calculations

Lifetime utility is

$$\begin{aligned}
& u(c_i(z_0)) + \sum_{j=i+1}^{I} \left(\beta_i \times ... \times \beta_{j-1}\right) \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) u(c_j(z^{0+j-i})) \\
= \; & u(c_i(z_0)) + \sum_{j=i+1}^{I} \left(\beta_i \times ... \times \beta_{j-1}\right) \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) u\left(\left[R^{j-i} \left(\beta_i \times ... \times \beta_{j-1}\right)\right]^{\frac{1}{\gamma}} c_i(z_0)\right) \\
= \; & \left(1 + \sum_{j=i+1}^{I} \left(\beta_i \times ... \times \beta_{j-1}\right) \sum_{z^{0+j-i}} \pi_0(z^{0+j-i}|z_0) \left(\left[R^{j-i} \left(\beta_i \times ... \times \beta_{j-1}\right)\right]^{\frac{1}{\gamma}}\right)^{1-\gamma}\right) \frac{\left(\chi^i LTI^i_{z_0}\right)^{1-\gamma}}{1-\gamma} \\
= \; & \left(1 + \sum_{j=i+1}^{I} \left(\beta_i \times ... \times \beta_{j-1}\right) \left[\left(R^{j-i}\right)^{\frac{1-\gamma}{\gamma}} \left(\beta_i \times ... \times \beta_{j-1}\right)^{\frac{1-\gamma}{\gamma}}\right]\right) \frac{\left(\chi^i LTI^i_{z_0}\right)^{1-\gamma}}{1-\gamma}
\end{aligned}$$

Note that lifetime utility depends on the initial state z_0 and the initial start of period wealth distribution $A^i W$ only through the term $LTI^i_{z_0}$.

What is the welfare cost of entering in a recession? Define it as the solution ω to

$$\begin{aligned} & \left(1+\sum_{j=i+1}^{I}(\beta_i\times...\times\beta_{j-1})\left[\left(R^{j-i}\right)^{\frac{1-\gamma}{\gamma}}(\beta_i\times...\times\beta_{j-1})^{\frac{1-\gamma}{\gamma}}\right]\right)\frac{\left(\chi^i LTI^i_{z_H}\right)^{1-\gamma}}{1-\gamma}(1+\omega)^{1-\gamma} \\ = & \left(1+\sum_{j=i+1}^{I}(\beta_i\times...\times\beta_{j-1})\left[\left(R^{j-i}\right)^{\frac{1-\gamma}{\gamma}}(\beta_i\times...\times\beta_{j-1})^{\frac{1-\gamma}{\gamma}}\right]\right)\frac{\left(\chi^i LTI^i_{z_L}\right)^{1-\gamma}}{1-\gamma} \end{aligned}$$

or

$$\begin{aligned} \left(LTI^i_{z_H}\right)^{1-\gamma}(1+\omega)^{1-\gamma} &= \left(LTI^i_{z_L}\right)^{1-\gamma} \\ \log\left(LTI^i_{z_H}\right)+\log(1+\omega) &= \log\left(LTI^i_{z_L}\right) \\ \omega &\approx \log\left(\frac{LTI^i_{z_L}}{LTI^i_{z_H}}\right) \end{aligned}$$

This entity is straightforward to calculate. In particular, for scenario 1 (no asset price recession) the calculation is

$$\omega_1 \approx \log\left(\frac{A^i_{-1}W_{-1}+\varepsilon_i(z_L)(1-\theta)z_L+\sum_{j=i+1}^{I}\frac{1}{R^{j-i}}\sum_{z^{0+j-i}}\pi_0(z^{0+j-i}|z_L)\varepsilon_i(z\left(z^{0+j-i}\right))(1-\theta)z\left(z^{0+j-i}\right)}{A^i_{-1}W_{-1}+\varepsilon_i(z_H)(1-\theta)z_H+\sum_{j=i+1}^{I}\frac{1}{R^{j-i}}\sum_{z^{0+j-i}}\pi_0(z^{0+j-i}|z_H)\varepsilon_i(z\left(z^{0+j-i}\right))(1-\theta)z\left(z^{0+j-i}\right)}\right)$$

For scenario 2 (asset price recession) the calculation is

$$\omega_2 \approx \log\left(\frac{A^i_{R}W_{R}+\varepsilon_i(z_L)(1-\theta)z_L+\sum_{j=i+1}^{I}\frac{1}{R^{j-i}}\sum_{z^{0+j-i}}\pi_0(z^{0+j-i}|z_L)\varepsilon_i(z\left(z^{0+j-i}\right))(1-\theta)z\left(z^{0+j-i}\right)}{A^i_{-1}W_{-1}+\varepsilon_i(z_H)(1-\theta)z_H+\sum_{j=i+1}^{I}\frac{1}{R^{j-i}}\sum_{z^{0+j-i}}\pi_0(z^{0+j-i}|z_H)\varepsilon_i(z\left(z^{0+j-i}\right))(1-\theta)z\left(z^{0+j-i}\right)}\right)$$

We can also translate the answers into dollar numbers, by computing $\omega LTI^i_{z_H}$.

J The Economy with Housing

Residential real estate is the single most important component of household net worth. We now extend our baseline model to allow households to invest in three assets: stocks, bonds, and housing. The key finding will be that under a particular rescaling of parameter values, the welfare consequences of recessions in the economies with and without housing are identical. In what follows, the superscript H is used to differentiate parameter values from their counterparts in the original model without housing.

Preferences are given by

$$E\left[\sum_{i=1}^{I}\prod_{j=1}^{i}\beta_j^H\,\frac{x_i^{1-\gamma^H}-1}{1-\gamma^H}\right],$$

where x_i is a composite consumption bundle comprising nondurable consumption c and housing services s, with respective shares υ^H and $1-\upsilon^H$:

$$x_i = c_i^{\upsilon^H} s_i^{1-\upsilon^H}.$$

This Cobb-Douglas specification is consistent with extensive empirical evidence (see, for example, Davis and Ortalo-Magne 2011). We assume that the aggregate supply of housing services is state invariant and normalized to one. Housing is perfectly divisible, and there is a frictionless rental market. Thus, agents can separate the decisions of how much housing to consume versus how much housing to own for investment purposes. The technology for producing the nondurable good c is exactly the same as in the baseline model, and this technology produces z^H units of nondurable output. Now corporate debt is a promise to deliver one unit of the *composite* good x in the next period. The firm issues B^H units of this debt each period. We then have the following proposition.

Proposition 2. *If*

$$\begin{aligned}
(1-\theta) &= (1-\theta^H)\upsilon^H,\\
B &= \left(1+\frac{(1-\upsilon^H)}{\upsilon^H\theta^H}\right)B^H,\\
\{z\} &= \left\{\left(z^H\right)^{\upsilon^H}\right\},\\
\{\beta_i\} &= \left\{\beta_i^H\right\},\\
\{\varepsilon_i\} &= \left\{\varepsilon_i^H\right\},\\
\gamma &= \gamma^H,\\
\Gamma &= \Gamma^H,
\end{aligned}$$

then in the economy without housing, the life-cycle consumption profiles and the law of motion *for wealth are identical, state by state, to their counterparts in the economy with housing:*

$$
\begin{aligned}
c_i(z, A, a) &= x_i(z^H, A, a) \\
G(z, A, z') &= G^H(z^H, A, (z^H)').
\end{aligned}
$$

where G^H denotes the law of motion for wealth shares in the model with housing. It follows immediately that the welfare consequences of recessions in the two economies are identical for each age, in each aggregate state.

The key to this result is that, in this model, rents comove perfectly with output, and house prices comove perfectly with the value of nonhousing wealth. Let $p^h(z, A)$ denote the ex-rent price of housing, and let rents and the price of non-durable consumption be denoted by $r(z, A)$ and $p^c(z, A)$. Then in the model with housing,

$$
\begin{aligned}
p^h(z.A) &= \frac{(1-\upsilon^H)}{\upsilon^H \theta^H}\left[p(z, A) + q(z, A)B^H\right] \\
\frac{r(z, A)}{p^c(z, A)} &= \frac{(1-\upsilon^H)}{\upsilon^H} z^H.
\end{aligned}
$$

Thus, the housing asset offers the same returns as the market portfolio of corporate equity and debt. It follows that introducing housing does not affect households' ability to share risks across generations. At the same time, given Cobb-Douglas preferences and the implied constant expenditure shares for nondurable consumption and housing, introducing housing does not change the shapes of the life-cycle profiles for consumption or asset holdings.

Proof. In order to prove the previous result we begin by describing the decision problem in the model with housing. In a series of steps, we will then show that this problem is isomorphic to the decision problem in the model without housing.

Let $y_i(z, A, a)$ and $\lambda_i^e(z, A, a)$, $\lambda_i^h(z, A, a)$ denote the optimal household policy functions for total savings and for the fraction of savings invested in equity and housing. Let $c_i(z, A, a)$, $s_i(z, A, a)$, and $a_i'(z, A, a, z')$ denote the policy functions for nondurable consumption, housing consumption, and for shares of next period wealth. Let $p^c(z, A)$, $p^h(z, A)$, $p(z, A)$, $q(z, A)$, and $r(z, A)$ denote, respectively, the price of the nondurable consumption good, the price of housing, the price of stocks, the price of bonds, and the rental rate for housing, all relative to the composite good x.

The dynamic programming problem of the household reads as

$$v_i(z, A, a) = \max_{c,y,\lambda^e,\lambda^h,a'} \left\{ u\left(c, s\right) + \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'} \; v_{i+1}\left(z', A', a'\right) \right\} \qquad \text{s.t.} \tag{A-29}$$

$$p^c(z,A)c + y + r(z,A)s = \varepsilon_i(z)w(z) + W(z,A)a \tag{A-30}$$

$$a' = \frac{\left(\lambda^e \frac{[p(z',A')+d(z')]}{p(z,A)} + \lambda^h \frac{\left[p^h(z',A')+r(z',A')\right]}{p^h(z,A)} + (1-\lambda^e-\lambda^h)\frac{1}{q(z,A)}\right) y}{W(z',A')} \tag{A-31}$$

$$A' = G(z,A,z'). \tag{A-32}$$

The aggregate value of start-of-period wealth in the model with housing is the value of aggregate payments to asset holders in the period $d(z) + r(z, A) + B$, plus the ex-dividend value of equity and housing $p(z, A) + p^h(z, A)$. Thus,

$$W(z', A') = p(z', A') + d(z') + p^h(z', A') + r(z', A') + B.$$

A recursive competitive equilibrium can be defined as in the baseline model.

1. Result on Rents: The agent's first-order condition with respect to the consumption of housing services implies

$$s_i(z, A, A_i) = \frac{(1-\upsilon)}{\upsilon r(z,A)} c_i(z, A, A_i) p^c(z, A). \tag{A-33}$$

Summing across age-groups,

$$\sum_{i=1}^{I} s_i(z, A, A_i) = \frac{(1-\upsilon)p^c(z,A)}{\upsilon r(z,A)} \sum_{i=1}^{I} c_i(z, A, A_i).$$

Imposing market clearing gives

$$r(z, A) = \frac{(1-\upsilon)}{\upsilon} p^c(z, A) z. \tag{A-34}$$

Let e denote total expenditure in units of the composite good:

$$e_i(z, A, A_i) = p_c(z, A)c_i(z, A, A_i) + r(z, A)s_i(z, A, A_i).$$

Substituting in A-33 gives

$$e_i(z, A, A_i) = \frac{1}{\upsilon}p_c(z, A)c_i(z, A, A_i). \tag{A-35}$$

Define aggregate consumption/output as

$$X(z, A) = \sum_i x_i(z, A, A_i).$$

In equilibrium

$$\begin{aligned} x_i(z, A, A_i) &= c_i(z, A, A_i)^{\upsilon}\left(\frac{(1-\upsilon)}{\upsilon r(z, A)}c_i(z, A, A_i)p^c(z, A)\right)^{1-\upsilon} \\ &= c_i(z, A, A_i)z^{\upsilon-1}. \end{aligned} \tag{A-36}$$

so aggregate composite consumption (aggregate output) is

$$\begin{aligned} X(z, A) &= \sum_i x_i(z, A, A_i) \\ &= z^{\upsilon-1}\sum_i c_i(z, A, A_i) \\ &= z^{\upsilon}. \end{aligned}$$

Solve for $p^c(z, A)$ by setting the value of aggregate composite consumption equal to aggregate expenditure (recall $p^x(z, A)$ is normalized to one):

$$\begin{aligned} z^{\upsilon} &= \sum_i e_i(z, A, A_i) \\ &= \frac{1}{\upsilon}p_c(z, A)z. \end{aligned}$$

Thus,

$$p^c(z, A) = \upsilon z^{\upsilon-1}. \tag{A-37}$$

It follows that

$$r(z,A) = \frac{(1-\upsilon)}{\upsilon} z p^c(z,A) = (1-\upsilon) z^{\upsilon}.$$

2. Result on House Prices: Recall that the numeraire here is the composite consumption good. Thus, dividends are given by

$$\begin{aligned} d(z,A) &= p^c(z,A)\theta z - B + q(z,A)B \\ &= \upsilon\theta z^{\upsilon} - B + q(z,A)B. \end{aligned}$$

where the second line follows from equation (A-37). Consider the following two assets: a claim to aggregate capital income (unlevered equity) and housing. The respective returns to the two assets are

$$\begin{aligned} \frac{p(z',A') + d(z',A') + B}{p^u(z,A)} &= \frac{p^u(z',A') + \upsilon\theta\,(z')^{\upsilon}}{p^u(z,A)} \\ \frac{p^h(z',A') + r(z',A')}{p^h(z.A)} &= \frac{p^h(z',A') + (1-\upsilon)\,(z')^{\upsilon}}{p^h(z.A)}. \end{aligned}$$

Note that the income streams associated with these two assets are in fixed proportions. It follows immediately that

$$p^h(z.A) = \frac{(1-\upsilon)}{\upsilon\theta} p^u(z,A).$$

3. Result on Portfolio Choice: Given that the return to housing is equal to the return to unlevered equity, we can write the law of motion for individual wealth as

$$\begin{aligned} a'W(z',A') &= \left(\lambda^e \frac{[p(z',A') + d(z')]}{p(z,A)} + \lambda^h \frac{[p(z',A') + d(z') + B]}{p(z,A) + q(z,A)B} + (1-\lambda^e-\lambda^h)\frac{1}{q(z,A)}\right) y \\ &= \left(\left(\lambda^e + \frac{\lambda^h p(z,A)}{p(z,A) + q(z,A)B}\right) \frac{[p(z',A') + d(z')]}{p(z,A)} + \left((1-\lambda^e-\lambda^h) + \frac{\lambda^h q(z,A)B}{p(z,A) + q(z,A}\right.\right. \\ &= \left(\widetilde{\lambda}\frac{[p(z',A') + d(z')]}{p(z,A)} + \left(1-\widetilde{\lambda}\right)\frac{1}{q(z,A)}\right) y, \end{aligned}$$

where

$$\widetilde{\lambda} = \lambda^e + \frac{\lambda^h p(z,A)}{p(z,A) + q(z,A)B}.$$

Note that (i) there is no reference to house prices or rents in this law of motion, and (ii) there is only one meaningful portfolio choice for agents, given that the return to housing is a linear combination of the returns to equity debt.

4. Expression for Aggregate Wealth: Aggregate wealth can be written as

$$\begin{aligned} W(z,A) &= p(z,A) + d(z) + p^h(z,A) + r(z,A) + B \\ &= p(z,A) + q(z,A)B + p^h(z,A) + \upsilon\theta z^\upsilon + (1-\upsilon)z^\upsilon \\ &= p(z,A) + q(z,A)B + \frac{(1-\upsilon)}{\upsilon\theta}\left(p(z,A) + q(z,A)B\right) + \left(\upsilon\theta + 1 - \upsilon\right) z^\upsilon. \end{aligned}$$

Define

$$\begin{aligned} \tilde{p}(z,A) &= \left(1 + \frac{(1-\upsilon)}{\upsilon\theta}\right) p(z,A) \\ \tilde{B} &= \left(1 + \frac{(1-\upsilon)}{\upsilon\theta}\right) B \\ \tilde{d}(z,A) &= \left(1 + \frac{(1-\upsilon)}{\upsilon\theta}\right) \upsilon\theta z^\upsilon - \tilde{B} + q(z,A)\tilde{B}. \end{aligned}$$

In terms of this notation, aggregate wealth is given by

$$W(z,A) = \tilde{p}(z,A) + q(z,A)\tilde{B} + \left(1 + \frac{(1-\upsilon)}{\upsilon\theta}\right)\upsilon\theta z^\upsilon. \quad \text{(A-38)}$$

5. Final Household Problem: We can now write the agent's problem without any reference to housing as

$$v_i(z,A,a) = \max_{c,y,\widetilde{\lambda},a'} \left\{ \frac{x^{1-\gamma}}{1-\gamma} + \beta_{i+1} \sum_{z' \in Z} \Gamma_{z,z'}\ v_{i+1}\left(z', A', a'\right) \right\}$$

subject to

$$x + y = \varepsilon_i(z)(1-\theta)\upsilon z^\upsilon + W(z,A)a \quad \text{(A-39)}$$

$$a' = \frac{\left(\widetilde{\lambda}\frac{\left[\tilde{p}(z',A')+\left(1+\frac{(1-\upsilon)}{\upsilon\theta}\right)\upsilon\theta(z')^{\upsilon}-\tilde{B}+q(z',A')\tilde{B}\right]}{\tilde{p}(z,A)}+\left(1-\widetilde{\lambda}\right)\frac{1}{q(z,A)}\right)\ y}{W(z',A')}, \tag{A-40}$$

taking as given laws of motion for z and A, and where $W(z', A')$ is given by A-38. It is clear that this model is identical to the model without housing defined in the text, as long as parameter values in the model without housing are the following functions of parameters in the model with housing, where the latter are now denoted with superscript H :

$$\begin{aligned} 1-\theta &= (1-\theta^H)\upsilon^H \\ z &= \left(z^H\right)^{\upsilon} \\ B &= \left(1+\frac{(1-\upsilon^H)}{\upsilon^H\theta^H}\right)B^H. \end{aligned}$$

□