On the Existence, Uniqueness, and Computability of Non-Optimal Recursive Equilibria in Linear Quadratic Economies

Sebastián Ceria and José-Víctor Ríos-Rull [1]

September 26, 1994

[1]Address: Graduate School of Industrial Administration, Carnegie Mellon University, Schenley Park Pittsburgh, PA, 15213 and Department of Economics, University of Pennsylvania, 3718 Locust Walk, Philadelphia, PA, 19104.
Conversations with Finn Kydland, Ed Prescott, Mike Trick, and Ellen McGrattan have been very fruitful in the development of this research, and are acknowledged.

Abstract

For economies that either have many agents or types of agents or for which the first welfare theorem does not hold, the number of equilibria is generally unknown. A large class of interesting economies can be studied through linear-quadratic approximations in a recursive framework. In this paper we derive a set of necessary conditions that equilibrium laws of motion have to satisfy. One of them is that certain components of equilibrium economy-wide laws of motion have to satisfy a second order polynomial in matrices. We describe the known properties of these polynomials and how to find all their solutions, and, therefore, all candidates for equilibria. The number of solutions is generically finite. These candidates provide a procedure to obtain the other objects that constitute an equilibrium. Only those candidates that succeed in this task are equilibria, the rest are disregarded. This procedure returns both an answer to the issue of how many equilibria exist and to the problem of their computation.

1 Introduction

Equilibria in economies with an infinite number of periods is sometimes defined recursively. This restricts attention to Markov characterizations of its properties, without the need of keeping track of the whole history of the economy. Prescott & Mehra (1980) showed that in representative agent economies where the welfare theorems hold, this notion of equilibrium corresponds to the Arrow-Debreu notion. Furthermore, it relates to the Arrow-Debreu concept in very much the same fashion that Bellman's equation relates to the standard social planner's problem: it gives the same answer in a much more tractable way. This tractability amounts basically to computability. In economies where the welfare theorems hold and there is a representative agent, competitive equilibria are very easy to find as there is a unique Pareto optimum: the solution to the social planner's problem. This problem can typically be approximated adequately and its solution found through numerical methods. For the class of economies for which either equilibria are not optima or there are multiple agents or types of agents and, therefore, multiple Pareto optima, this approach cannot be followed, and equilibria has to be computed directly.

In this paper we study the issues of existence and number of equilibria, and their computability in linear quadratic economies. We derive necessary conditions that equilibrium laws of motions have to satisfy. These amount to solve certain second order polynomial in matrices. They also provide a procedure to solve for them. The solutions to the polynomial provide us with candidates for equilibrium laws of motions. After this is done, we utilize other necessary conditions to rule out some solutions to the polynomial as equilibrium laws of motion. Those solutions that pass all these hurdles constitute **all** the recursive equilibria. In other words, we describe a set of procedures that can be guaranteed to find all equilibria under standard assumptions for convergence of Newton type of algorithms for all except a measure zero set of economies.

Computability of equilibria is another very problematic issue. Two different approaches have been typically used in the literature on linear-quadratic economies.[1] The first defines an operator whose fixed point is a recursive equilibrium. This operator defines an iterative procedure. In practice, these iterations usually converged to a fixed point (see Kydland (1989a) and Hansen & Prescott (1994) for details). The second approach is McGrattan (1991). She takes advantage of the procedure described by Vaughan (1970) to compute

[1]Marcet (1989) parameterizes expectations in order to solve for equilibria in nonlinear models. His method could also be applied for the linear case.

solutions to the Ricatti equation of a standard maximization problem. Vaughan notes that the system matrix for the backward canonical equation has eigenvalues constituting reciprocal pairs. McGrattan constructs a similar matrix to the one used by Vaughan. Under certain conditions on the eigenvalues of this matrix, Vaughan's procedure can be used to compute the law of motion of the economy. None of these methods guarantee ex-ante that the procedures involved deliver an equilibrium with probability one (assuming of course that equilibria exist), although they typically do.

Economies of the first type include for example Kydland & Prescott (1977), Kydland & Prescott (1980), Cooley & Hansen (1988), Braun (1989), Kydland (1989b), McGrattan (1989), and Chang (1990). Ríos-Rull (1994), and Ríos-Rull (1993), have economies with a large number of agents; some of their equilibria are optima, some are not, but all of them have to be computed directly. In these economies existence of equilibria is proved through a variety of arguments (see Coleman (1991) and Ríos-Rull (1992)), but uniqueness is specially problematic. Sufficient conditions for uniqueness are given in Coleman (1991) and Greenwood & Huffman (1992) for specific type of distorsions in the growth model. The kind of problems that can appear in more general models is illustrated in Kehoe, Levine, & Romer (1989).

The paper is organized as follows. First, a simple model that can be used to illustrate this issues is described as it is the notion of recursive equilibria. In Section 3, the current procedures to compute equilibria are briefly described. Section 4 contains the description of the necessary conditions that equilibrium laws of motion have to satisfy. They include the fact that a subset of them have to be roots of a second order polynomial. In Section 5, the knowledge about solutions to second order polynomials is described. We also include a description of the procedures to find them. In Section 6, we describe the next step, which is to check whether those solutions do in fact turn out to be equilibria or not. This amounts to checking for feasibility and for whether individual decision rules obtained by taking solutions to the polynomials as laws of motions of the economy, do in fact generate those laws of motions. In Section 7, an example is used to illustrate the procedures. Section 8 concludes.

2 The Model

The model we use is the simplest economy we can think of. The model assumes that the state variables are already determined. It is a representative agent economy. We will follow the notation of Hansen & Prescott (1994). We will refer to next period's variables by primes. We choose the convention of referring to vectors as column vectors. We use the superindex

^{T} to denote the transpose. Let the exogenous state variables be labeled by $z \in \mathbf{R^{n_z}}$. These typically include an exogenous shock (productivity, government policies, and others), and (for reasons having to do with approximation methods) it usually also contains a constant. The endogenous economy-wide state variables are $S \in \mathbf{R^{n_s}}$, while the individual state variables are $s \in \mathbf{R^{n_s}}$. Let

$$(z^T, S^T, s^T, S'^T, s'^T)\, R \begin{pmatrix} z \\ S \\ s \\ S' \\ s' \end{pmatrix}$$

denote the current return of the agent as a function of todays exogenous state z, endogenous aggregate state S, individual state s, tomorrow's aggregate state S', and tomorrow's individual state s'. For simplicity we assume that the agent chooses tomorrow's individual state s'. So in effect, the controls are tomorrows individual's states. R denotes a square matrix of dimension $n_z + 4n_s$ that determines the current return. For simplicty, we assume that the agent faces no restriction. Preferences of the agent are the expected discounted sum (with discount factor β) of the current returns for each period. This choice of model is just to simplify notation. We could differentiate explicitly between control variables, and next period's state variables, and add as restrictions the law of motion of endogenous state variables as linear functions of current states and controls. This could reduce the computational burden of some problems at the expense of some extra notation. From our point of view it is better to reduce notation. Multiple agent models are outside our formulation but it is immediate to notice that the same type of necessary conditions for equilibria that arise below arise in such models.

Agents take laws of motion of the aggregate state as given and choose s'. In equilibrium, $s = S$, but when agents solve their individual problem they think of themselves as uncapable of effectively affecting the value of S. Therefore, they take the law of motion of aggregate variables S, $S' = D\binom{z}{S}$ as given. We are now ready to define equilibrium

Definition 1 *A Recursive Competitive equilibrium for this quadratic economy is a set of matrices $V_{\{n_z+2n_s\}\times\{n_z+2n_s\}}$, the value function of the agent, $D_{\{n_s\}\times\{n_z+n_s\}}$, the law of motion for the aggregate endogenous states and $d_{\{n_s\}\times\{n_z+2n_s\}}$, the individual decisions such that:*

- *The agent optimizes, i.e. d, solves the problem of the agent, given laws of motion for the aggregate states z, and S, which in turn generates value function v,*

$$(z^T, S^T, s^T)\, v \begin{pmatrix} z \\ S \\ s \end{pmatrix} = \max_{s'} \quad (z^T, S^T, s^T, S'^T, s'^T)\, R \begin{pmatrix} z \\ S \\ s \\ S' \\ s' \end{pmatrix}$$

$$+\beta \quad E\Big\{(z'^T, S'^T, s'^T)\, v \begin{pmatrix} z' \\ S' \\ s' \end{pmatrix} |z\Big\} \tag{1}$$

$$s.t. \qquad z' = F\, z + \epsilon \tag{2}$$

$$S' = D \begin{pmatrix} z \\ S \end{pmatrix} \tag{3}$$

- *The agent is representative, i.e.*

$$D \begin{pmatrix} z \\ S \end{pmatrix} = d \begin{pmatrix} z \\ S \\ S \end{pmatrix} \tag{4}$$

Here v is the value function that has to be found. R is a symmetric matrix and concave in s and s'. Matrix F is the law of motion of the exogenous state variables. Matrix D gives the law of motion of the economy. A solution to this problem delivers not only the value function, represented by matrix v, but also, optimal decision rules for s', namely $d\left(\begin{smallmatrix} z \\ S \\ s \end{smallmatrix}\right)$. Of course, we are not interested in solutions for arbitrary laws of motion for the economy, but only in the equilibrium ones. These are those laws of motion for aggregate variables that generate individual decisions of agents which in turn have to generate the aggregate law of motion of the economy, $D\left(\begin{smallmatrix} z \\ S \end{smallmatrix}\right) = d\left(\begin{smallmatrix} z \\ S \\ S \end{smallmatrix}\right)$.

This problem is standard **given** the law of motion of the economy, D. In the class of representative agent economies that satisfy the First Welfare Theorem, there is a candidate for matrix D, the solution to the social planner's problem. However in the class of non-optimal economies, D is unknown, and has to be found simultaneously with matrices v and d.

3 How Equilibrium is Found

In this section a brief description of the standard methods used to compute equilibria without the aid of the welfare theorems is given. A more detailed exposition can be found in Hansen & Prescott (1994).

The standard procedure to obtain a solution to this problem is to iterate in the following way:

Step 1. Starting with a negative semidefinite matrix v^0.

Step 2. Define matrix $\hat{v}^0$ in the following way:

$$(z^T, S'^T, s'^T)\hat{v}^0 \begin{pmatrix} z \\ S' \\ s' \end{pmatrix} \equiv E\{(z'^T, S'^T, s'^T)v^0 \begin{pmatrix} z' \\ S' \\ s' \end{pmatrix} |z\}$$

where $z' = F\,z + \epsilon$, with ϵ being independent of z, and having variance-covariance matrix Σ_ϵ.

Step 3. Solve the problem:

$$\max_{s'} \quad (z^T, S^T, s^T, S'^T, s'^T)R \begin{pmatrix} z \\ S \\ s \\ S' \\ s' \end{pmatrix} + \beta\,(z^T, S'^T, s'^T)\hat{v}^0 \begin{pmatrix} z \\ S' \\ s' \end{pmatrix} \tag{5}$$

when solving this problem, S' is taken as given. A solution to this problem is a linear function:

$$s' = \hat{d}^1 \begin{pmatrix} z \\ S \\ s \\ S' \end{pmatrix}$$

Step 4. Using the fact that the representative agent is truly representative, i.e. that in equilibrium $s = S$, and $s' = S'$ the following linear equation is satisfied:

$$S' = \hat{d}^1 \begin{pmatrix} z \\ S \\ S \\ S' \end{pmatrix} = (\hat{d}^1_z, \hat{d}^1_S, \hat{d}^1_s, \hat{d}^1_{S'}) \begin{pmatrix} z \\ S \\ S \\ S' \end{pmatrix} = \hat{d}^1_z\,z + (\hat{d}^1_S + \hat{d}^1_s)\,S + \hat{d}^1_{S'}\,S' \tag{6}$$

note that we decompose $\hat{d}^1$ into submatrices indexed by the variables they are associated with. Then, we obtain D^1 by solving the linear system,

$$D^1 = (I - \hat{d}^1_{S'})^{-1} \begin{pmatrix} \hat{d}^1_z \\ \hat{d}^1_S + \hat{d}^1_s \end{pmatrix} \tag{7}$$

Step 5. Obtain individual decision rule d^1, by

$$d^1 \begin{pmatrix} z \\ S \\ s \end{pmatrix} = [(\hat{d}^1_z, \hat{d}^1_S) + \hat{d}^1_{S'} D^1] \begin{pmatrix} z \\ S \end{pmatrix} + \hat{d}^1_s(s) \tag{8}$$

Step 6. Substitute in (5) S' and s' for its values in terms of D^1 and d^1, and obtain a new matrix v^1.

Step 7. Repeat steps 2 to 6 with the new matrix v^1.

This procedure continues until a matrix v that returns itself is found.

This procedure has no guarantee of convergence. In fact, Step 3 requires the inversion of a matrix, and this is not necessarily always possible. There is also no knowledge of how many matrices v return themselves when using this procedure.

4 A Necessary Condition for Equilibria

In this section we derive a necessary condition for the economy-wide equilibrium law of motion of the economy. A part of this law of motion has to satisfy a second order polynomial in matrices. In order to be able to say more about how the aggregate law of motion of the economy D relates to the fundamentals, which are given by matrices R, and F, and parameter β, we start by noting that the first order and envelope conditions of problem (1) are:

$$R_{s'.} \begin{pmatrix} z \\ S \\ s \\ S' \\ s' \end{pmatrix} + \beta R_{s.} \begin{pmatrix} Fz \\ S' \\ s' \\ S'' \\ s'' \end{pmatrix} = 0 \tag{9}$$

where $R_{s'.}$ and $R_{s.}$ are the submatrices of R that contain rows s', and s. Recall that we are searching for solutions D and d that respectively provide the economy-wide law of motion, and the individual decision rules, as functions of the state variables. Then, these solutions have to satisfy,

$$R_{s'}.\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ D_z & D_S & 0 \\ d_z & d_S & d_s \end{pmatrix}\begin{pmatrix} z \\ S \\ s \end{pmatrix} + \beta R_{s.}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ D_z & D_S & 0 \\ d_z & d_S & d_s \end{pmatrix}\begin{pmatrix} F & 0 & 0 \\ D_z & D_S & 0 \\ d_z & d_S & d_s \end{pmatrix}\begin{pmatrix} z \\ S \\ s \end{pmatrix} = 0 \tag{10}$$

where next period's states have been substituted using D and d. This can be rewritten as

$$\begin{aligned}(R_{s'z}, R_{s'S}, R_{s's})\begin{pmatrix} z \\ S \\ s \end{pmatrix} + (R_{s'S'}, R_{s's'})\begin{pmatrix} D_z & D_S & 0 \\ d_z & d_S & d_s \end{pmatrix}\begin{pmatrix} z \\ S \\ s \end{pmatrix} \\ +\beta(R_{sz}, R_{sS}, R_{ss})\begin{pmatrix} F & 0 & 0 \\ D_z & D_S & 0 \\ d_z & d_S & d_s \end{pmatrix}\begin{pmatrix} z \\ S \\ s \end{pmatrix} \\ +\beta(R_{sS'}, R_{ss'})\begin{pmatrix} D_z & D_S & 0 \\ d_z & d_S & d_s \end{pmatrix}\begin{pmatrix} F & 0 & 0 \\ D_z & D_S & 0 \\ d_z & d_S & d_s \end{pmatrix}\begin{pmatrix} z \\ S \\ s \end{pmatrix} = 0\end{aligned} \tag{11}$$

Recall that in equilibrium, the individual agent is representative, so $D_z(z) + D_S(S) = d_z(z) + d_S(S) + d_s(S)$, so we can rewrite the previous equation as

$$\begin{aligned}(R_{s'z}, R_{s'S} + R_{s's})\begin{pmatrix} z \\ S \end{pmatrix} + (R_{s'S'} + R_{s's'})(D_z\ D_S)\begin{pmatrix} z \\ S \end{pmatrix} \\ +\beta(R_{sz}, R_{sS} + R_{ss})\begin{pmatrix} F & 0 \\ D_z & D_S \end{pmatrix}\begin{pmatrix} z \\ S \end{pmatrix} \\ +\beta(R_{sS'} + R_{ss'})(D_z\ D_S)\begin{pmatrix} F & 0 \\ D_z & D_S \end{pmatrix}\begin{pmatrix} z \\ S \end{pmatrix} = 0\end{aligned} \tag{12}$$

Note that this has to be true for all possible pairs (z, S), therefore, the following two equations have to hold:

$$\begin{aligned}(R_{s'S} + R_{s's}) + [(R_{s'S'} + R_{s's'}) + \beta(R_{sS} + R_{ss})]D_S \\ +\beta(R_{sS'} + R_{ss'})D_S D_S &= 0\end{aligned} \tag{13}$$

$$\begin{aligned}\{(R_{s'S'} + R_{s's'}) + \beta[(R_{sS} + R_{ss}) + (R_{sS'} + R_{ss'})D_S]\}D_z \\ +(R_{s'z} + \beta R_{sz}F) + \beta(R_{sS'} + R_{ss'})\, D_z\, F &= 0\end{aligned} \tag{14}$$

Note that the first of the last two equations is a second order polynomial in the square matrix D_S. In the next section we describe our knowledge about such polynomials. Specifically, we describe properties relating its coefficient matrices (which are all known functions of the fundamentals of our economies) to the roots. Note also that once D_S is found, the second equation is a linear system in D_z of order $n_z \times n_s$, which is readily solvable, and it has a unique solution except on a linear subspace.

5 Solving matrix polynomials

In this section we study second order matrix polynomials of the form:

$$P(D_S) = AD_S^2 + BD_S + C = 0 \tag{15}$$

as the one in (13). Here D_S is an $n \times n$ matrix of unknowns, and A, B, C are real matrices in $\mathbf{R}^{n_S \times n_S}$. The matrix D_S that satisfies $P(D_S) = 0$ is called a *solvent.* This problem was studied by Davies (1979) in his doctoral thesis, and in more general terms by Dennis, Traub, & Weber (1976, 1978).

Associated with equation (15) we have the scalar problem of finding $\lambda \in \mathbf{R}$ such that:

$$det(P(\lambda I)) = det(A\lambda^2 I + B\lambda I + C) = 0 \tag{16}$$

which appears in the literature as the *quadratic eigenvalue problem.* A scalar λ that satisfies $det(P(\lambda I)) = 0$ is called a *latent root* of $P(\lambda I)$. A vector b such that $P(\lambda I)b = 0$ is called the *latent vector* associated to λ. It is easy to see that $P(\lambda I)$ has at most $2n$ latent roots, and, in fact, less whenever A is singular.

One of the main difficulties in solving $P(D_S) = 0$ is that there is no generalization of the Fundamental Theorem of Algebra to the case of matrix polynomials. It is possible to have an infinite number of solutions to equation (15), or none at all (see, for example Davies (1979)). In the next section we study some of the properties of solvents. In particular, we give conditions under which a matrix polynomial has a finite number of solvents. These conditions are satisfied for a dense set of coefficient matrices and amount to a certain derived matrix being non-singular.

5.1 Properties of solvents

We now detail some of the known results for matrix polynomials and properties of their solutions.

It is not difficult to show that the n eigenvalues of D_S, a solvent of $P(D_S) = 0$ are all latent roots of $P(\lambda I)$.

The following theorem in Lancaster (1966) gives a sufficient condition for the existence of solvents of $P(D_S) = 0$.

Theorem 5.1 *If $P(\lambda I)$ has n linearly independent latent vectors, $b_1, \ldots, b_n$, corresponding to latent roots $\delta_1, \ldots, \delta_n$, then $Q\Lambda Q^{-1}$ is a solvent, where $Q = [b_1, \ldots, b_n]$ and Λ is a diagonal matrix with elements $\delta_1, \ldots, \delta_n$.*

The last theorem is important because it gives a basic algorithm for finding solutions to the matrix polynomial (15). Note that latent roots can be found by solving a polynomial of degree $2n$ and latent vectors by solving a system of linear equations. We now show that under certain conditions all solutions to the matrix polynomial have to be of the form given in Theorem 5.1.

Theorem 5.2 *If a solvent of $P(D_S) = 0$ is diagonalizable, then it must be of the form $Q\Lambda Q^{-1}$, where Λ is a diagonal matrix formed by latent roots of $P(\lambda I)$ and Q is a matrix formed by the associated latent vectors.*

Theorem 5.2 is characterizes all the solvents of $P(D_S) = 0$ whenever we know that the solvents D_S is diagonalizable. An important class of diagonalizable matrices is the case of symmetric real matrices. In this case, not only the matrix is diagonalizable but also it has all real eigenvalues. Whenever a matrix is not symmetric but has all different eigenvalues, either real or complex, the matrix is also diagonalizable, although one may have to work in the field of complex numbers for achieving such representations. For more conditions on diagonalizable matrices see (Golub & Van Loan, 1989). An important property about diagonalizable matrices is that they are dense in $C^{n\times n}$, that is, given a matrix which is not diagonalizable we can find another matrix which is arbitrarily close to it (in some matrix norm) and is diagonalizable. This implies that, unless for some unusual cases, all the solvents have to be of the form described in Theorem 5.2, i.e. they can be obtained by a diagonal matrix formed by latent roots of the matrix polynomial premultiplied by the matrix of the corresponding latent vectors and post-multiplied by the inverse of this matrix. It also gives a fundamental property of the number of solutions of $P(D_S) = 0$.

Corollary 5.3 *If $P(\lambda)$ has $2n$ distinct latent roots, and every set of n latent vectors are linearly independent, then there are exactly $\binom{2n}{n}$ different solvents of $P(D_S) = 0$.*

5.2 Sensitivity Analysis

One of the most important properties of matrix polynomials is the fact that a small change in the data (the matrices A, B, C) can lead to important changes in the solvents of $P(D_S)$, not only in terms of the distance from one solution to the other, but also in terms of the number of solutions of $P(D_S) = 0$. In his doctoral thesis, Davies (1979) shows an example where a small change in a coefficient of the matrix A changes the number of solutions of the matrix polynomial from none to infinity. He also gives, however, a condition under which small changes in the data can only cause small changes in the solution. This condition corresponds to checking that a certain matrix is non-singular. Whenever this matrix is singular small changes in the data can lead to unbounded changes in the solution. It should be noted, that the set of singular matrices has zero measure, and, for this reason, the cases of extreme sensitivity should not be expected in practice.

Given an $n \times n$ matrix D_S, the function $P(D_S)$ yields another $n \times n$ matrix. The derivative of $P(D_S)$ (called the Frechet derivative) should be thought of as an $n^2 \times n^2$ matrix, defined more precisely by the following formula:

$$P'(D_S) = (AD_S + B) \otimes I + A \otimes D_S^T$$

where $Y \otimes Z$ is a block matrix with its (i, j) block equal to $Y_{ij}Z$.

Davies (1979) showed that whenever $P'(D_S)$ is singular for every D_S small changes in the data can lead to unbounded changes in the solvents. As noted before, this problem should not be expected in practice, since the set of data matrices A, B, C for which $P'(D_S)$ is singular for every D_S has zero measure.

5.3 Algorithms for matrix polynomials

Davies (1983) describes an algorithm for solving the matrix equation $P(D_S) = 0$. The subroutine, called SQUINT (Solving the Quadratic by Iterating Newton Triangularizations), is publicly available in Morris (1990).

After an initial guess is chosen, the iterates are generated by a form of Newton's method for solving systems of nonlinear equations. Specific properties of the problem are used to simplify the computation of the Newton iterates. This method has been proven very effective in practice. Under some standard assumptions for convergence of Newton type algorithms, the method is guaranteed to converge to a solution of the problem. To get all possible solutions this method needs to be reapplied with different initial guesses. Numerical experiments show that when starting with sufficiently different initial guesses convergence to all the different solvents is usually achieved. See Davies (1983) for more details on the implementation and numerical experience with SQUINT.

6 What Solutions are Equilibria

The solutions to the polynomial equation (13) D_S, and associated linear equation (14) D_z, that are found through the methods described in the previous section satisfy necessary conditions for being recursive equilibria. These necessary conditions are not sufficient, however. There are other conditions that have to be satisfied by (D_S, D_z) in order to be an equilibrium. They are feasibility, and the fact that they have to generate well defined individual problems for the agents which in turn generate (D_S, D_z) when aggregated.

With feasibility, we mean properties like bounded paths in economies that have bounded production possibilities which disregard as equilibrium all unbounded paths. This, however should be taken with care, as linear quadratic economies are typically approximations to more general economies and we should be careful that the criteria on boundedness is not a feature that arises only because of the approximation.

The definition of Recursive Equilibria requires two objects, the aggregate law of motion D, and the individual decision rules d. The decision rules are obtained when solving the individual problem of the agent (1), subject to (2), and (3). The solutions to the polynomial that generate feasible paths can then be used to find candidates for D. In order to obtain a solution, the candidate law of motion has to induce a well defined maximization problem. This means that not all solutions to the polynomial will necessarily be apt to the task. Those that do provide us with a pair (D, d) candidate for equilibria. There is only one condition left that has to be satisfied: aggregation, namely that $D_z z + D_S S = d_z z + d_S S + d_s S$.

All these arguments amount to the following. An equilibria is a pair (D, d) where D satisfies (13) and (14), and where d solves (1) given (2) and (3) (when D is used in (3)). Generically, there is at most a finite number of such equilibria. There exist computational procedures that permit to answer the question how many are there, and how to find them.

7 An Example: A Simple Growth Economy

This is the economy used by Long & Plosser (1983). It is the simplest case we can think of. It has a unique optimal competitive equilibrium that can be found by solving the social planner's problem. Furthermore, the solution is the same than that of a loglinear approximation, therefore, we can compare the outcome of our procedures with the true equilibrium.

Agents care about a discounted stream of utility from the consumption good only. Technology can be represented by a constant returns to scale production function that takes labor and capital and produces output that can be used for consumption or capital next period. There is full depreciation.

We can write the recursive notion of equilibrium by noting that k, the stock of wealth, is the aggregate state. The individual state will be the agent's own wealth a. Then the state vector is $(z, S, s) = (1, k, a)$. The current return of the problem is, $U(f(k)+(a-k)f'(k)-a')$. To compute matrix R, we first have to parameterize the economy. Let $f(k) = k^{.36}$. Let $\beta = .96$, and let $U(c) = log\, c$. The steady state of this economy is $\bar{k} = .190117$. With these parameters, it is easy to comput R, the quadratic approximation around the steady state, where the approximation is made in logs, which is:

$$R = \begin{pmatrix} -1.4407 & .72492E-06 & .42941 & .0 & -.69279 \\ .72492E-06 & .17604 & -.17604 & .0 & -0.10847E-06 \\ .42941 & -.17604 & .12374 & .0 & .14527 \\ .0 & .0 & .0 & .0 & .0 \\ -.69279 & -0.10847E-06 & .14527 & .0 & -.40352 \end{pmatrix}$$

There are two values of D_S and D_z that solve (13) and (14). They are $log\, k' = 3.0359 + 2.8287 log\, k$, and $log\, k' = -1.0623 + .35975 log\, k$. Candidates for equilibria have to satisfy (13) and (14), but this is only necessary. In particular, out of our two candidates, we can see that the first has the property that for all initial conditions above certain minimal level, the limit stock of capital is infinite as the economy grows without bound. This, however, is unfeasible: the production technology has an upper bound. This rules out the first law of motion for capital as a recursive equilibria.

Alternatively, we can verify whether the D_S and D_z that we found generate well defined problems for the agents, and whether their optimal decision rules $s' = d_z z + d_S S + d_s s$ do in fact generate D_z and D_S. In the Long and Plosser economy, the problem of the agent is not well defined when the law of motion for the aggregate capital stock used is the first one. Iterations on it do not converge, as the associated operator is not a contraction. When the aggregate law of motion used is the second one, the individual decision problem converges to $s' = -1.0882\, z - .63953\, S + 1.0002\, s$. This is quite similar to the path given by the second law of motion for the aggregate economy.

In this example we have computed the two possible solutions to the polynomial in matrices associated to the recursive equilibria. Then we check whether these solutions satisfy the other necessary conditions for equilibria. We saw that one solution fails both criteria while the other passes both, remaining as the unique candidate for equilibrium.

8 Conclusion

In this paper we have studied equilibria in general linear-quadratic economies. These economies might have multiple equilibria. We derive a set of necessary conditions that equilibrium laws of motion have to satisfy. Among them, they have to satisfy a polynomial in matrices. We describe procedures to find all the solutions of this polynomial, and, therefore, all candidates for equilibria. This set is generically finite. Given a solution we can try to obtain the rest of the objects that are required by the definition of recursive competitive equilibrium. In the cases that this can be done, it can be checked whether we in fact have an equilibrium. This provides a criteria to determine whether an equilibrium exists, whether it is unique or not, and procedures to find **all** equilibria.

References

Braun, A. R. (1989). The dynamic interaction of distortionary taxes and aggregate variables in post-war u.s. data. Ph.D. Dissertation. Carnegie Mellon University.

Chang, L. J. (1990). Business cycles with distortionary taxes and inflation. Ph.D. Dissertation. Carnegie Mellon University.

Coleman, J. W. I. (1991). Equilibrium in a production economy with an income tax. *Econometrica*, *59*(1091-1104).

Cooley, T. F. & Hansen, G. D. (1988). The inflation tax in a real business cycle model. *American Economic Review*, *21*, 247–80.

Davies, G. J. (1979). *Numerical Solution of a Quadratic Matrix Equation.* Ph.D. thesis, University of New Mexico, Albuquerque, NM.

Davies, G. J. (1983). An algorithm to compute solvents of the matrix equation $ax^2+bx+c=0$. *ACM Transactions on Mathematical Software*, *9*(2), 246–254.

Dennis, J. E., Traub, J. F., & Weber, R. P. (1976). The algebraic theory of matrix polynomials. *SIAM J. Numer. Anal.*, *13*(6), 831–845.

Dennis, J. E., Traub, J. F., & Weber, R. P. (1978). Algorithms for solvents of matrix polynomials. *SIAM J. Numer. Anal.*, *15*(3), 523–533.

Golub, G. & Van Loan, C. (1989). *Matrix Computations* (Second edition). John Hopkins University Press.

Greenwood, J. & Huffman, G. W. (1992). On the existence and uniqueness of nonoptimal equilibria in dynamic stochastic economies. Federal Reserve Bank of Minneapolis, Staff Report 151.

Hansen, G. D. & Prescott, E. C. (1994). Recursive methods for computing equilibria of business cycle models. Forthcoming as Chapter 2 of *Frontier of business Cycles Research*, ed. T. Cooley.

Kehoe, T., Levine, D., & Romer, P. (1989). Characterizing equilibria of models with externalities and taxes as solutions to optimization problems. unpublished manuscript, Federal Resereve Bank of Minneapolis.

Kydland, F. E. (1989a). Monetary policy in models with capital. In van der Ploeg, F. & de Zeeuw, A. J. (Eds.), *Dynamic Policy Games in Economics*. North Holland, Amsterdam.

Kydland, F. E. (1989b). The role of money in a business cycle model. Institute for Empirical Macroeconomics. Discussion Paper 23.

Kydland, F. E. & Prescott, E. C. (1977). Rules rather than discretion: the inconsistency of optimal plans. *Journal of Political Economy*, *85*(3), 473–91.

Kydland, F. E. & Prescott, E. C. (1980). Dynamic optimal taxation, rational expectations and optimal control. *Journal of Economic Dynamics and Control*, *2*, 79–91.

Lancaster, P. (1966). *Lambda-Matrices and vibrating Systems*. Pergamon.

Long, J. B. & Plosser, C. (1983). Real business cycles. *Journal of Political Economy*, *91*, 1345–1370.

Marcet, A. (1989). Solving non-linear stochastic models with parameterizing expectations. Unpublished Manuscript, Carnegie Mellon University.

McGrattan, E. R. (1989). The macroeconomic effects of tax policy in an equilibrium model. Unpublished manuscript, Department of Economics, Duke University.

McGrattan, E. R. (1991). A note on computing equilibria in linear models. Duke University Working Paper.

Morris, A. H. (1990). *NSWC Library of Mathematic Subroutines*. Naval Surface Warfare Center, Dahlgren, Va.

Prescott, E. C. & Mehra, R. (1980). Recursive competitive equilibrium: the case of homogeneous households. *Econometrica*, *48*(6), 1356–79.

Ríos-Rull, J.-V. (1992). Equivalence between standard and recursive definitions of equilibria. Unpublished Manuscript, Carnegie-Mellon University.

Ríos-Rull, J.-V. (1993). On the quantitative importance of market completeness. Unpublished Manuscript, University of Pennsylvania.

Ríos-Rull, J.-V. (1994). Life cycle economies and aggregate fluctuations. Unpublished Manuscript.

Vaughan, D. R. (1970). A nonrecursive algebraic solution for the discrete riccati equation. *IEEE Transactions on Automatic Control*, *AC-15*, 597–99.