Constrained Efficient Allocations in the Bewley-İmrohoroglu-Huggett-Aiyagari Model

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Introduction

- The equilibrium steady state of this model has the property that aggregate capital $\mathbf{K^A}$, is such that its marginal productivity is lower than the rate of time preference $\mathbf{1 + r^A < 1/\beta}$, there is more capital than in the steady state of an economy without shocks.

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- There are two parts to our inquiry

  1. How to ask the question of what is the right amount of capital GIVEN the frictions of the model.

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  2. What is the answer

- Pure theoretical question. (But quantitative, too.)

Let's see the logic in a 2-Period model

- Workers are the same in 1st period: income y, and make consumption-savings decision.

- In $t = 2$, random labor endowment: e_1 (prob π), $e_2 > e_1$ (prob $1 - \pi$).

- Law of large numbers: total labor, L, in period 2 is

$$\pi e_1 + (1 - \pi) e_2.$$

- Neoclassical production using CRS f in period 2. Prices r and w.

- No insurance markets: just "precautionary" savings with capital.

Definition 1 A CE is a vector (K, r, w) such that

- *K solves*

$$\max_{k\in[0,y]} u(y-k)+\beta\left\{\pi u(rk+we_1)+(1-\pi)u(rk+we_2)\right\}$$

- *$r = f_k(K, L)$ and $w = f_l(K, L)$, with $L = \pi e_1 + (1-\pi)e_2$.*

- Equilibrium utility as a function of K:

$$u(y-K)+\beta\left\{\pi u(f_k(K,L)K+f_l(K,L)e_1)\right.$$
$$\left.+(1-\pi)u(f_k(K,L)K+f_l(K,L)e_2)\right\}.$$

- Question here: is there a $\hat{K}$ that beats equilibrium K?

- I.e., is the equilibrium *constrained efficient?* Note that markets *remain incomplete*!

No: constrained inefficiency!

- Equilibrium K is too high.

- Intuition: a lower K
  - **raises r and lowers w, thus decreasing the de-facto risk**
  - **while only distorting behavior given prices in a second-order way.**

- Lesson: when markets are incomplete, a planner should take into account, and alter, how consumers influence prices!

- Connection: incomplete-markets GE literature (Diamond, Stiglitz, Hart, Geanakoplos, Mas-Colell, Cass, Polemarchakis, Drèze, Magill, Quinzii, . . .). No good examples though!

- So there is NOT a constrained form of the First Welfare Theorem.

3-Period model

- In a 3rd period, it is not clear that raising r is a good thing:
- It helps those with high capital, i.e., those who were lucky in period 2! This makes insurance worse.
- So equilibrium K could also be *too low*!
- Another issue: in period 2, *who* should save?
- Quantitative analysis needed.

The Bewley-İmrohoroglu-Huggett-Aiyagari Model

- A continuum of agents.
- Idiosyncratic shocks to eff labor $e_i \in E\{e_1, \cdots, e_i, \cdots, e_I\}$, i.i.d across agents. Markov $\Gamma_{e,e'}$.
- Standard preferences: $E_0 \left\{\sum_t \beta^t u(c_t)\right\}$
- Agents cannot violate a borrowing constraint (or a no default constraint) $a \geq \underline{a}$. For now assume $a \in A = [0, \overline{a}]$. More later.

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- State of economy is x, a measure on $S = E \times A$.
- Aggregation: $K = \int_S a \, dx, \qquad L = \int_S e \, dx$, constant.
- Prices $r = r(K) = f_K(K, L) - \delta, \quad w = w(K) = f_L(K, L)$.
- Budget constraint $c + a' = a\,(1 + r) + e\,w$.

The Consumer Problem is (recursively, if all goes well)

$$v(x,e,a) \;=\; \max_{c,a' \in A} \; u\,(c) + \; \beta \sum_{e'} \; \Gamma_{e,e'} \quad v(x',e',a') \qquad \text{s.t.}$$

$$c + a' = a\;[1 + r\,(x)] + e\; w\,(x)\,; \qquad x' = H(x)$$

with solution $a' = h(x, e, a)$. The first order condition is

$$u_c\left(a\;[1 + r\,(x)] + e\; w\,(x) - a'\right) \geq \; \beta \sum_{e'} \Gamma_{e,e'} \;\; v_3(x',e',a')$$

with equality if $a' > \underline{a}$. The envelope condition is

$$v_3(x,e,a) = [1 + r(x)] \;\; u_c\left(a\;[1 + r\,(x)] + e\; w\,(x) - a'\right)$$

Compactly we can write

$$v(x,e,a) \;=\; \max_{c,a'\in A} \; u\,(c) + \;\beta \sum_{e'} \;\Gamma_{e,e'} \;\; v(H(x),e',a') \qquad \text{s.t.}$$

$$c + a' = a\;[1 + r\,(x)] + e\;w\,(x)\,;$$

with foc

$$u_c\,(x,e,a,h(x,e,a)) \geq \;\beta \sum_{e'} \Gamma_{e,e'} \;\; [1 + r(H(x))]$$
$$u_c\left\{H(x), e', h(x,e,a), h[H(x), e', h(x,e,a)]\right\}$$

Equilibrium

With h and Γ, we construct an individual transition process Q by

$$Q(x, e, a, B; h) = \sum_{e' \in B_e} \Gamma_{ee'} \ \chi_{h(x,e,a) \in B_a}$$

where χ is the indicator function.

Equilibrium requires that H is generated by h.

$$x'(B) \ = H(x)(B) \ = \int_S Q(x, e, a, B; h) \ dx = T(x, h)$$

A steady state is $\widetilde{x}$ such that $\qquad \widetilde{x} = T(\widetilde{x}, h)$.

Optimal Allocations

- Are there better allocations that can be implemented **without** violating the frictions of the model? (inexistence of state contingent trades and the existence of borrowing limits)

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- Are there better allocations that can be implemented **without** violating the frictions of the model? (inexistence of state contingent trades and the existence of borrowing limits)

- There may be too much or too little capital. Once the first welfare theorem does not hold there is no reason to believe that the prices are the correct ones.

How to assess allocations

- To find optimal allocations given the model's frictions we solve an equal weight social planner's problem that incorporates the fact that agents cannot insure themselves and that are subject to borrowing constraints.

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- This means that the planner cannot reallocate resources among agents but can change the problem that they solve to account for the (forgive me the expression) pecuniary externality or price effects.

Planner's Problem

$$\Omega(x) \quad = \quad \max_{y(x,e,a)\geq \underline{a}} \int_S u\left[a(1+\ r(x)) + e\ w(x) - y\right]\ \ dx \quad + \beta\, \Omega(x')$$

$$\text{s.t.} \qquad\qquad x' = T[x,y] \qquad\qquad \text{with foc:}$$

$$0 \ \geq\ -u_c\left(a\ [1+r(x)] + e\ w(x) - y(x,e,a)\right)\ \ x(de,da)\ + \beta \sum_{e'}$$

$$\Gamma_{e,e'}\ \ u_c\left(y(x,e,a)\ \left[1+r\left(x'\right)\right] + e'\ w\left(x'\right) - y(x',e',y(x,e,a)]\right)\ \ x(de,da)$$

$$+\,\beta\ \ x(de,da) \int_S \left[e'\ f_{LK}(K',L)) + a'\ f_{KK}(K',L)\right]$$

$$u_c\left(a'\ \left[1+r\left(x'\right)\right] + e'\ w\left(x'\right) - y[x',e',a']\right)\ \ dx' \qquad\qquad \forall e,a\in S$$

Compactly

$$
\begin{aligned}
0 \quad \geq \quad & -u_c\,(x, e, a, y(x, e, a)) \\
& + \;\beta \sum_{e'} \Gamma_{e,e'} \;\; u_c \left(x', e', y(x, e, a), y[x, e', y(x', e, a)]\right) + \\
& \qquad\qquad \beta \int_S \left[e' \; f_{LK}(K', L) + a' \; f_{KK}(K', L)\right] \\
& \qquad\qquad\qquad u_c \left(a' \; \left[1 + r\left(x'\right)\right] + e' \; w\left(x'\right) - y\left[x', e', a'\right]\right) \; dx'
\end{aligned}
$$

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0 \;\geq\; & -u_c\left(x, e, a, y(x, e, a)\right) \\
& + \beta \sum_{e'} \Gamma_{e,e'} \; u_c\left(x', e', y(x, e, a), y[x, e', y(x', e, a)]\right) + \\
& \qquad \beta \int_S \left[e' \; f_{LK}(K', L) + a' \; f_{KK}(K', L)\right] \\
& \qquad\quad u_c\left(a' \; \left[1 + r\left(x'\right)\right] + e' \; w\left(x'\right) - y\left[x', e', a'\right]\right) \; dx'
\end{aligned}
$$

and Most Compactly

$$
0 \;\geq\; -u_c + \beta \sum_{e'} \Gamma_{e,e'} \; u'_c \; + \beta \int_S \left[e' \; w'_K + a' \; r'_K\right] \; u'_c \; dx'
$$

A Few Remarks

1. The third term is the sum of tomorrow's changes in income induced by additional savings weighted by the marginal utility (hence the consumption poor matter more).

2. In a representative agent model the third term is zero, and hence the equilibrium is optimal (2nd Welfare theorem). A representative agent model collapses the integral with respect to wealth yielding

$$u_c \geq \beta \left(1 + r'\right) \sum_{e'} \Gamma_{e,e'} \; u'_c \; + \; \beta \sum_{e'} \Gamma_{e,e'} \; \left[L' \; f'_{LK} + K' \; f'_{KK} \right] \; u'_c .$$

   The terms in braces are zero by the Euler theorem since the production function is homogeneous of degree 0.

3. The sign of the last term depends on the sign of the term in braces for the high marginal utility (poor) agents. The persistence of earnings determines how wealth and earnings determine poverty. Effectively, the model picks what poor means by its choice of low consumption: it chooses low consumption for those agents who are likeliest to hit the non-negativity asset constraint in the future. If poor agents have labor intensive income relative to the economy as a whole then the term in braces is positive and because of the Euler theorem so is the whole third term.

4. Consequently the issue of whether there is too much capital or too little capital in this economy depends on whether the poor agents' income is labor intensive or capital intensive. If it is labor intensive, they would benefit from more capital and the planner would like to have more capital than the market economy, and viceversa.

6. This is an empirical issue:

Let $e \in \{e_L - e_H\}$ be i.i.d. Let e_L be very unlikely and very small (i.e. unemployment). In this economy agents save to prevent that state. In this economy the poor are capital intensive, and hence the planner may want less capital than what the market allocates.

Alternatively, other model economies with more of a right tail of earnings will have wealthy people be capital rich (as more of our intuition says).

Other Interpretations

1. Imagine a government with an utilitarian objective that can enforce a tax code, yet its tax collectors are so incompetent that are incapacitated to extract resources. They would devise a highly non linear tax system that distinguishes between capital and labor income.

2. Imagine that you asked an agent if you were to be parachuted in a society, and if these societies were indexed by the savings functions of its agents, savings functions to which you will commit to, which society will make you choose. The agent will choose our decision rules.

The planner's St St: x^*, K^*, and $g^*(e,a)$ such that

- Stat Dbon, $\quad x^* = \lim_{n\to\infty} \; T^n(x, g^*)$.
- Aggr Capital, $K^* = \int \; a \; dx^*$ (with $r^* = r(K^*)$, $w^* = w(K^*)$).

• FOC: $\quad 0 \;\geq\; u_c \left[a\,(1+r^*) + e\,w^* - g^*\,(e,a)\right] +$

$$\beta \sum_{e'} \Gamma_{e,e'} \; u_c \left[g^*(e,a) \;\; (1+r^*) + e' \; w^* - g^*[e', g^*(e,a)]\right] + \;\; \beta$$

$$\int_S \left[e' \; w_K(K^*) + a' \; r_K(K^*)\right] \; u_c \left(a' \; [1+r^*] + e' \; w^* - g[e', a']\right) \; dx^*$$

A functional eqtn that can be solved by standard numerical methods.

A calibrated Economy From Diaz, Pijoan and Rios-Rull (03)

Earnings process of the high earnings variability economy

$e \in \{e_1, e_2, e_3\} =$	$\{1.00,$	$5.29,$	$46.55\}$
$\pi_{e,e'} =$	0.992	0.008	0.000
	0.009	0.980	0.011
	0.000	0.083	0.917
$\pi^\star =$	0.481	0.456	0.063

Calibration: the steady state of the market economy has an interest rate of about 4%. The capital output ratio of around 3 and labor share of 0.64 (which required $\beta = 0.887$, $\delta = 0.8$, $\theta = 0.36$). The intertemporal elasticity of substitution is set at 0.5.

Quantitative Properties: The Market Economy

The steady state of the market model economy

	Deterministic Ec.	Market Ec.
Aggregate Assets	1.736	4.016
Output	1.000	1.353
Capital Output ratio	1.736	2.969
Interest Rate (%)	12.740	4.124
Coeff. of Variation of Wealth	0.0	2.563
Gini Index of Wealth	0.0	0.861

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	Quintiles					*Top Groups (%)*		
Wealth Db	1st	2nd	3rd	4th	5th	5-10%	1-5%	0-1%
Market	0.00	0.00	1.45	3.39	95.16	25.38	38.00	14.55
US 98	–0.30	1.30	5.00	12.20	81.70	11.30	23.10	34.70

Large Precautionary Savings. Looks like a modern economy

Quantitative Properties: All Economies

The steady states of the calibrated economy

	Deterministic Economy	Market Economy	Planner Economy
Aggregate Assets	1.736	4.038	**14.742**
Output	1.000	1.355	2.160
Capital Output ratio	1.736	2.980	6.825
Interest Rate	12.740	4.081	-2.725
Coeff. of Variation of Wealth	0.0	2.543	2.549
Gini Index of Wealth	0.0	0.853	0.851

Enormous Changes.

The planner wants to save a lot more and increase wages.

Distribution

Wealth distribution in market and planner's economy

	Quintiles					*Top Groups (%)*		
Ec	1st	2nd	3rd	4th	5th	5-10%	1-5%	0-1%
Market	0.00	0.00	1.45	3.39	95.16	25.38	38.00	14.55
Planner	0.00	0.00	0.44	3.30	96.26	25.20	35.72	15.55
US 98	–0.30	1.30	5.00	12.20	81.70	11.30	23.10	34.70

Almost identical.

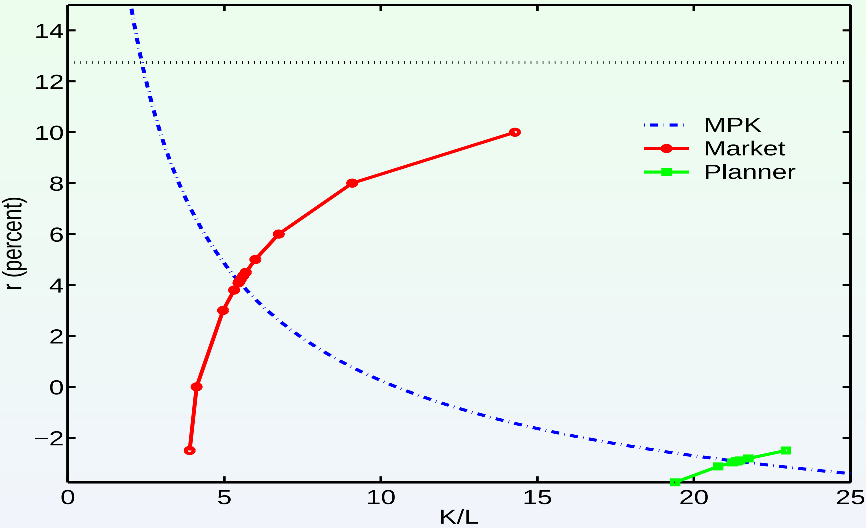

Figure: St St Supply and Demand of **K** for Market and Planner

Guess what, we do Transition dynamics

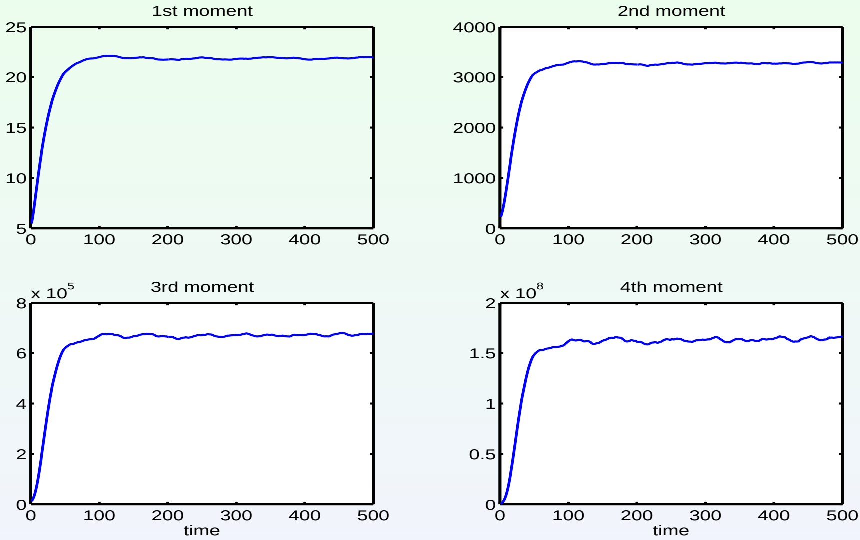

Figure: Path chosen by Planner Starting from the Steady State of the Market Ec

An Unemployment Economy

- Krusell Smith.
- $e_1 = 1$
- $e_2 = .05$. NOT a calibrated Economy.

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The steady states of the Unemployment model economy

	Deterministic Economy	Market Economy	Planner Economy
Aggregate Assets	2.959	3.373	3.288
Output	1.000	1.048	1.039
Capital Output ratio	2.959	3.217	3.166
Interest Rate	4.167	3.189	3.372
Coeff. of Variation of Wealth	0.0	0.199	0.195
Gini Index of Wealth	0.0	0.105	0.102

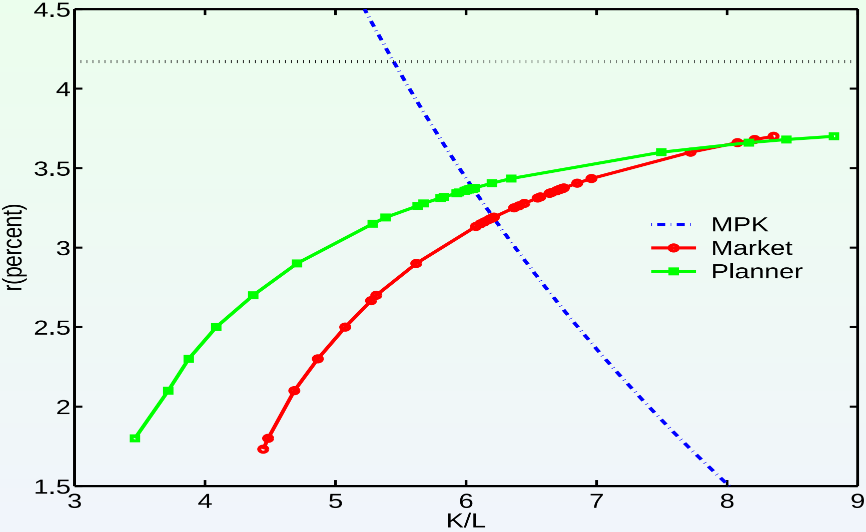


Figure: Steady-State Supply and Demand of Capital for the Market and Planner versions of the Unemployment Economy

But there are issues

- In the Market Economy there is a theorem that guarantees the existence of an upper bound.

- In the Planner's Economy there is no such thing. And sometimes there may be no upper bound. This MAY result in inexistence of Steady State.

Another Economy: The original Aiyagary Economy

General	β	σ	θ	δ
Parameters	0.96	2	0.36	0.08
Earnings	$e \in \{e_1, e_2, e_3\} =$	$\{.78,$	$1.00,$	$1.27\}$
	$\pi_{e,e'} =$	0.66	0.27	0.07
		0.28	0.44	0.28
		0.07	0.27	0.66
Stat. Dbon	$\pi^{\star} =$	0.337	0.326	0.337

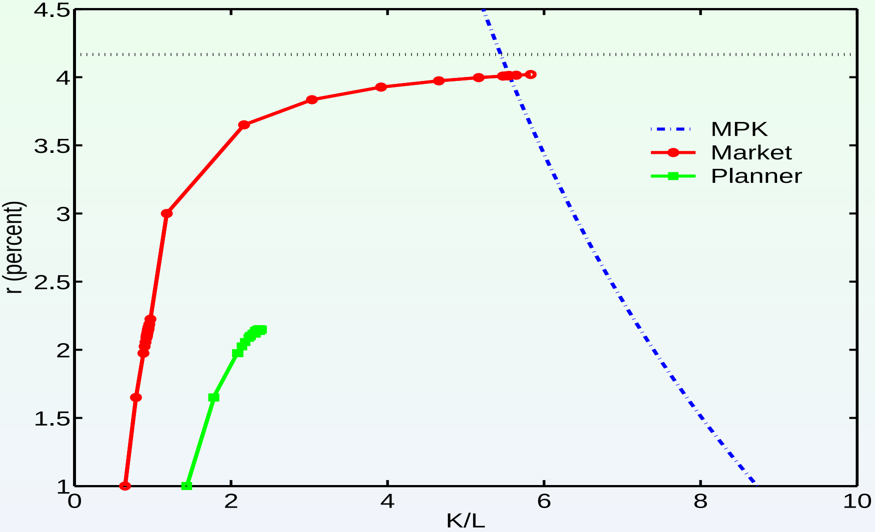


Figure: Steady State Supply and Demand of Capital for the ?: Inexistence.

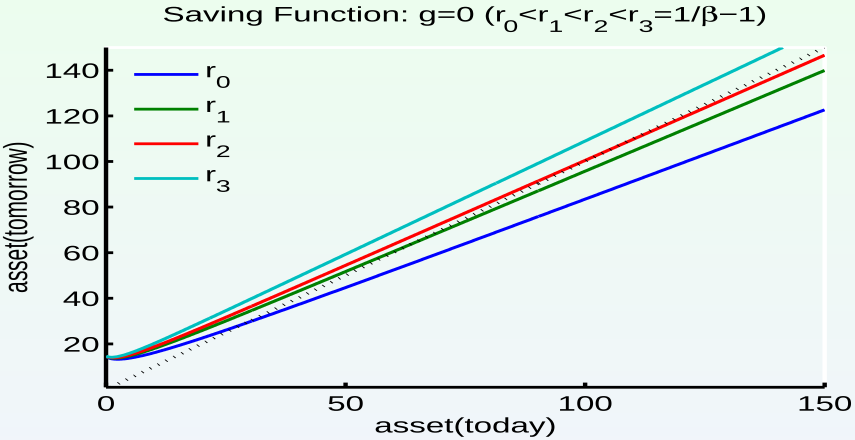


Figure: Planner's saving fn for e^3 as a function of r for given G.

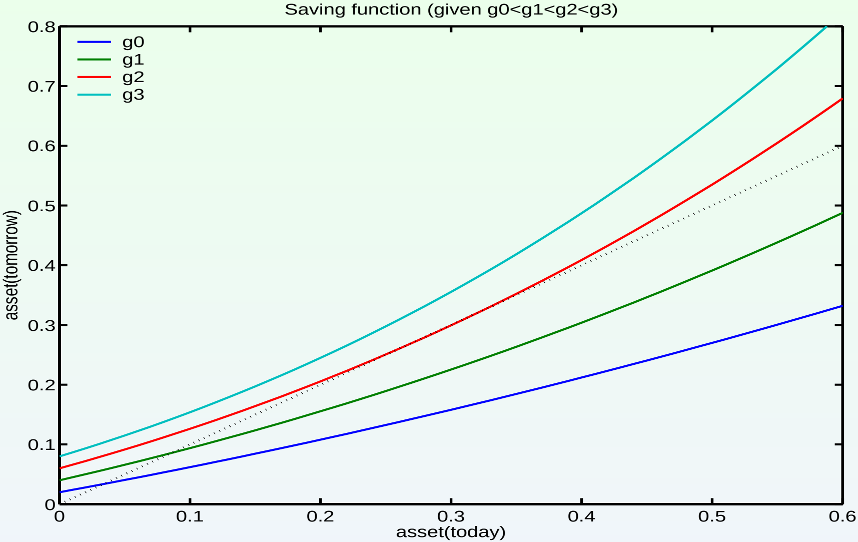


Figure: Planner's saving function for e^3 as a function of G for given r. For $\hat{G} > G2$ there is no Steady State.

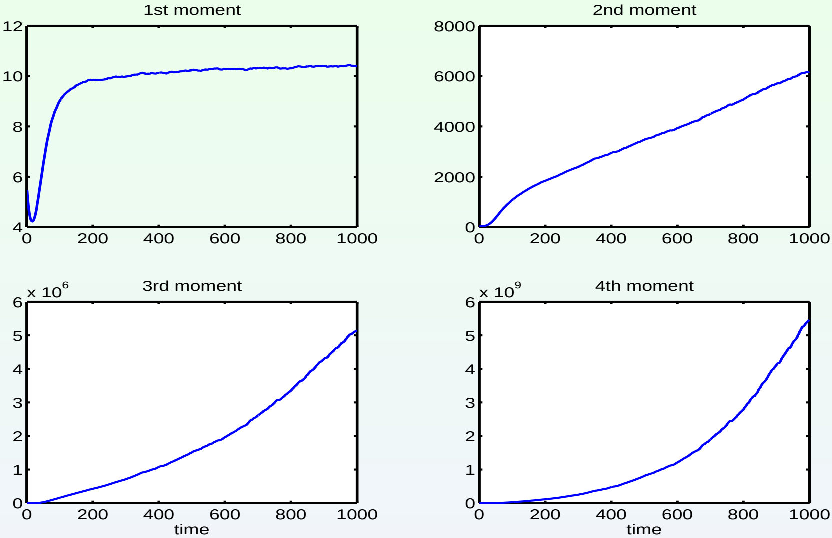
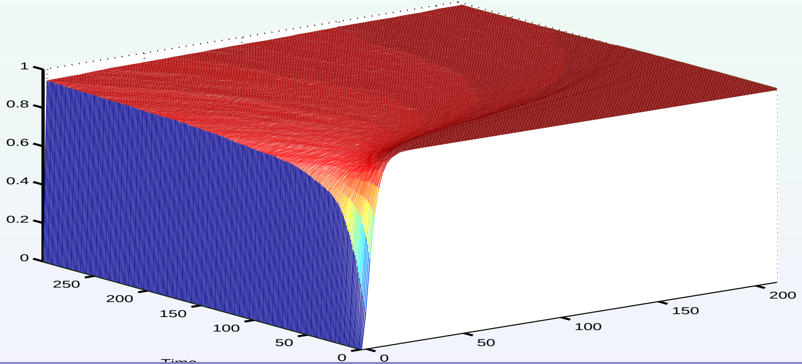


Figure: Transition from the Market Steady State : Aiyagari Model.


1
0.8
0.6
0.4
0.2
0
250
200
150
100
50
0
0
50
100
150
200

Conclusion

- We have asked how would the pecuniary externality of a planner that treats all agents the same and hence it cares more about the poor (Socialist?).

- We posted a dynamic programming problem that looks at this question.

- We have shown how the planner would choose things that yield a steady state that implies much larger capital than the market economy steady state.

- There are other (not calibrated economies that behave differently).
  - Planner chooses less Capital than market.
  - There is no steady state.