Calibration, Estimation, and Effects of Technology Shocks

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A Question

• What fraction of the variation in output and hours worked is due to technology shocks?
• This is a long-standing question in business cycle research, see, for instance, Kydland and Prescott (1982) and Fisher (2006).
• We’ll focus on hours worked.
Households

- There is a continuum of households solving the following problem

\[
\max \quad E_0 \left[ \sum_{t=0}^{\infty} \beta^t \left( \ln C_t - B \frac{H_t^{1+1/\nu}}{1 + 1/\nu} \right) \right] \tag{1}
\]

s.t.

\[
C_t + P_t^k X_t = W_t H_t + R_t^k P_t^k K_t \tag{2}
\]

\[
K_{t+1} = (1 - \delta) K_t + X_t \tag{3}
\]

- \(C_t\) is consumption, \(H_t\) is hours worked, \(X_t\) is investment (physical units), \(P_t^k\) is the price of the unit of the investment good (using the consumption good as numeraire), \(W_t\) is the wage, and \(R_t^k\) the rental rate of capital.
Households

- Labor supply

\[ H_t = \left( \frac{1}{B_t} \frac{W_t}{C_t} \right)^{\nu} \]

- Euler Equation

\[ 1 = \beta E_t \left[ \frac{P_{t+1}^k / C_{t+1}}{P_t / C_t} \right] ((1 - \delta) + R_{t+1}^k) \]
Firms

- Firms rent capital and labor services from Households and produce consumption and investment goods.

- Technology:

\[ C_t + \frac{X_t}{V_t} = A_t K_t^\alpha H_t^{1-\alpha} \]

- Profits:

\[ \Pi_t = C_t + P_t^k X_t - W_t H_t - R_t^k P_t^k K_t \]

- For the firms to be willing to produce both consumption and investment goods it has to be the case that \( P_t^k = 1/V_t \).

- The optimal choice of capital and labor implies

\[ W_t = (1 - \alpha) A_t K_t^\alpha H_t^{-\alpha}, \quad R_t^k P_t^k = \alpha A_t K_t^{\alpha-1} H_t^{1-\alpha} \]
NIPA and Exogenous Processes

• NIPA: investment is measured as $I_t = X_t P_t^k$. Hence,

$$Y_t = C_t + I_t = A_t K_t^\alpha H_t^{1-\alpha}.$$  

• Neutral technological shocks

$$A_t = \exp\{\gamma_a + \tilde{a}_t\} A_{t-1}, \quad \tilde{a}_t = \rho_a \tilde{a}_{t-1} + \sigma_a \epsilon_{a,t}$$  

• Investment-specific technology shocks

$$V_t = \exp\{\gamma_v + \tilde{v}_t\} V_{t-1}, \quad \tilde{v}_t = \rho_v \tilde{v}_{t-1} + \sigma_v \epsilon_{v,t}$$  

• To estimate the model, we make the preference shock time-varying

$$\ln(B_t/B) = \rho_b \ln(B_{t-1}/B) + \sigma_b \epsilon_{b,t}.$$
Equilibrium Conditions

- Endogenous variables: $Y_t$, $C_t$, $I_t$, $K_{t+1}$, $W_t$, $R^k_t$, $H_t$.
- The endogenous variables have to satisfy the following set of (nonlinear) rational expectations equations:

\[
H_t = \left( \frac{1}{B_t} \frac{W_t}{C_t} \right)^\nu
\]

\[
1 = \beta E_t \left[ \frac{C_t V_t}{C_{t+1} V_{t+1}} \left( (1 - \delta) + R^k_{t+1} \right) \right]
\]

\[
K_{t+1} = (1 - \delta)K_t + I_t V_t
\]

\[
Y_t = A_t K_t^\alpha H_t^{1-\alpha}
\]

\[
Y_t = C_t + I_t
\]

\[
W_t = (1 - \alpha) Y_t / H_t
\]

\[
R^k_t = \alpha Y_t / (P^k_t K_t)
\]
Detrending

- Along a balanced growth path the following variables are stationary

\[
\frac{Y_t}{Q_t}, \quad \frac{C_t}{Q_t}, \quad \frac{I_t}{Q_t}, \quad \frac{K_{t+1}}{Q_t V_t}, \quad \frac{W_t}{Q_t}, \quad R_t^k, \quad H_t.
\]

- where

\[
Q_t = A_t^{1-\alpha} V_t^{1-\alpha}.
\]

- We denote a detrended version of \( X_t \) by \( \hat{X}_t \).
The (detrended) endogenous variables have to satisfy the following set of (nonlinear) rational expectations equations:

\[
H_t = \left( \frac{1}{B_t \hat{C}_t} \right)^Λ
\]

\[
1 = \beta E_t \left[ \frac{\hat{C}_t Q_t V_t}{\hat{C}_{t+1} Q_{t+1} V_{t+1}} \left( (1 - \delta) + R_{t+1}^k \right) \right]
\]

\[
\hat{K}_{t+1} = (1 - \delta) \hat{K}_t \frac{Q_{t-1} V_{t-1}}{Q_t V_t} + \hat{I}_t
\]

\[
\hat{Y}_t = \hat{K}_t^\alpha \left( \frac{Q_{t-1} V_{t-1}}{Q_t V_t} \right)^\alpha H_t^{1-\alpha}, \quad \hat{Y}_t = \hat{C}_t + \hat{I}_t
\]

\[
\hat{W}_t = (1 - \alpha) \hat{Y}_t / H_t, \quad R_t^k = \alpha \frac{\hat{Y}_t}{\hat{K}_t} \frac{Q_t V_t}{Q_{t-1} V_{t-1}}
\]
Equilibrium Conditions

- Recall that
  \[ \frac{Q_t}{Q_{t-1}} = \exp \left\{ \frac{1}{1 - \alpha} (\gamma_a + \tilde{a}_t) + \frac{\alpha}{1 - \alpha} (\gamma_v + \tilde{v}_t) \right\}, \]
  \[ \frac{V_t}{V_{t-1}} = \exp \{ \gamma_v + \tilde{v}_t \} \]
- Define \( q_t = \frac{Q_t}{Q_{t-1}} \) and \( \nu_t = \frac{V_t}{V_{t-1}} \).
Equilibrium Conditions

\[ H_t = \left( \frac{1}{B_t \hat{C}_t} \hat{W}_t \right)^\nu, \quad \hat{K}_{t+1} = (1 - \delta) \hat{K}_t \frac{1}{q_t v_t} + \hat{I}_t \]
\[ 1 = \beta E_t \left[ \frac{\hat{C}_t}{\hat{C}_{t+1} q_{t+1} v_{t+1}} ((1 - \delta) + R_{t+1}^k) \right] \]
\[ \hat{Y}_t = \hat{K}_t^\alpha \left( \frac{1}{q_t v_t} \right)^\alpha H_t^{1-\alpha}, \quad \hat{Y}_t = \hat{C}_t + \hat{I}_t \]
\[ \hat{W}_t = (1 - \alpha) \hat{Y}_t / H_t, \quad R_t^k = \alpha \frac{\hat{Y}_t}{\hat{K}_t} q_t v_t \]
\[ q_t = \exp \left\{ \frac{1}{1 - \alpha} \left( \gamma_a + \tilde{a}_t \right) + \frac{\alpha}{1 - \alpha} \left( \gamma_v + \tilde{v}_t \right) \right\} \]
\[ v_t = \exp \{ \gamma_v + \tilde{v}_t \} \]
Solving the Model

- We can now calculate a steady state (in terms of the detrended variables), log-linearize the equilibrium conditions around the steady state, and apply a solution technique to solve the system of linear rational expectations difference equations.
- We show subsequently the relevant steady state ratios and log-linearized equations.
Steady States

\[ R^* = \frac{e^{(\gamma_a + \gamma_v)/(1 - \alpha)}}{\beta} - 1 + \delta \]

\[ K^* = \frac{\alpha e^{(\gamma_a + \gamma_v)/(1 - \alpha)}}{R^*} \]

\[ \frac{I^*}{Y^*} = \frac{K^*}{Y^*} \left[ 1 - (1 - \delta) e^{-(\gamma_a + \gamma_v)/(1 - \alpha)} \right] \]

\[ \frac{I^*}{K^*} = 1 - (1 - \delta) e^{-(\gamma_a + \gamma_v)/(1 - \alpha)} \]
Log-linearizations

\[ \tilde{H}_t = \nu (\tilde{W}_t - \tilde{C}_t - \tilde{B}_t) \]

\[ \tilde{K}_{t+1} = (1 - \delta) e^{-(\gamma_a + \gamma_v)/(1-\alpha)} \left[ \tilde{K}_t - \tilde{q}_t - \tilde{v}_t \right] + \frac{I^*}{K^*} \tilde{l}_t \]

\[ 0 = E_t \left[ \tilde{C}_t - \tilde{C}_{t+1} - (\tilde{q}_{t+1} + \tilde{v}_{t+1}) + \frac{R^*}{1 - \delta + R^*} \tilde{R}_{t+1} \right] \]

\[ \tilde{Y}_t = \alpha \tilde{K}_t + (1 - \alpha) \tilde{H}_t - \alpha [\tilde{q}_t + \tilde{v}_t] \]

\[ \tilde{Y}_t = \left( 1 - \frac{I^*}{Y^*} \right) \tilde{C}_t + \frac{I^*}{Y^*} \tilde{l}_t \]

\[ \tilde{W}_t = \tilde{Y}_t - \tilde{H}_t \]

\[ \tilde{R}_t^k = \tilde{Y}_t - \tilde{K}_t + \tilde{q}_t + \tilde{v}_t \]
Log-linearizations

\[
\begin{align*}
\tilde{q}_t &= \frac{1}{1-\alpha} \tilde{a}_t + \frac{\alpha}{1-\alpha} \tilde{v}_t \\
\tilde{a}_t &= \rho_a \tilde{a}_{t-1} + \epsilon_{a,t} \\
\tilde{v}_t &= \rho_v \tilde{v}_{t-1} + \epsilon_{v,t} \\
\tilde{B}_t &= \rho_b \tilde{B}_{t-1} + \epsilon_{b,t}
\end{align*}
\]
Answering the Question: Three Approaches

- A Calibration
- Bayesian estimation of the DSGE model
- A structural VAR, loosely based on the model
Data (To be Updated)

- Population (for conversion into per capita terms): total civilian noninstitutional (thousands, NSA); Source: DRI-Global Insight.
- Hours: Aggregate Hours Index (ID PRS85006033); Source: Bureau of Labor Statistics.
Data (To be Updated)

- According to our model

\[ Y_t = C_t + I_t = C_t + X_t P_t^k \]

where output, investment (and consumption) are measured in terms of consumption goods.

- In the data, we start from nominal output, consumption, and investment. Roughly:

\[ GDP_{nom} = C_{nom} + I_{nom} + G_{nom} + NetEX_{nom} \]

- We have to take a stand on what to do with \( G_{nom} \) and \( NX_{nom} \). How about: treating \( NX_{nom} \) as investment, splitting \( G_{nom} \) (attributing government expenditures on investment goods to investment and the remainder to consumption). What should we do with consumer durables?
After these adjustments we get

$$GDP^{nom} = \tilde{C}^{nom} + \tilde{I}^{nom}$$

which we obtain from the NIPA.

Using adjustments as above we can computed $\tilde{C}^{real}$ from NIPA.

Define a consumption deflator:

$$PCD = \tilde{C}^{nom} / \tilde{C}^{real}.$$ 

Then we can calculate real investment measured in terms of the consumption good, which is $I_t$ in the model, as $\tilde{I}^{nom}/PCD$. 
We can obtain $X_t$ in the model as $\tilde{I}^{nom}/(PCD \times P^k)$.

What remains to do:

- Capital Stock: Real capital in 1955; Source: Bureau of Economic Analysis, Fixed Asset Tables. Do we treat the real NIPA value as physical units ($K_0$ in our model)? Does it matter?

- Decide how to treat depreciation: depreciation rates versus real consumption of fixed capital (from Bureau of Economic Analysis (NIPA); we would need to convert this into consumption units).

- We need a discount factor $\beta$: compute averages of real interest rates to choose $\beta$. 
Calibration: Investment-specific Technology

- According to our model, the investment-specific technology shock corresponds to the relative price of investment goods, which we can measure in the data. Hence, we treat $V_t = 1/P_t^k$ as observed.
Calibration: Total Factor Productivity

- We compute the total factor productivity

\[ A_t = \frac{Y_t}{K_t^\alpha H_t^{1-\alpha}}, \]

which requires \( \alpha \) and \( K_t \).

- We can average data on the labor share \( W_t H_t / Y_t \) to obtain an estimate of \( \alpha \).

- Capital stock in period \( t = 0 \) is assumed to be in steady state. We then calculate a capital stock series recursively:

\[ K_{t+1} = (1 - \delta)K_t + I_t / P_t^k, \]

- Investment \( (I_t, \text{valued in terms of consumption goods}) \) is observed.

- We are using an average depreciation rate based on the Cummins-Violante depreciation series.
Calibration: Shock Processes

- Now that we have constructed estimates of $A_t$ and $V_t$ we can fit autoregressive processes.
- Using data from 1955 to 2006 we obtain the following point estimates
  \[
  \Delta \ln A_t = \Delta \ln A_{t-1} + 0.007 \tilde{\epsilon}_{A,t} \\
  \Delta \ln V_t = (1 - 0.8) \cdot 0.007 + 0.8 \Delta \ln V_{t-1} + 0.003 \tilde{\epsilon}_{V,t}
  \]
- We do not utilize the preference shock: $B_t = B$. 

Calibration

- We can calibrate $\beta$ based on observations on real interest rates.
- Traditional approach: link labor supply elasticity to steady state relationship. Suppose preferences are of the form

$$\ln C_t + \ln(1 - H_t)$$

Then Frisch elasticity is given by $$(1 - H^*)/H^*$$. If households work 1/3 of their time then Frisch elasticity is 2.
- We choose three values for $\nu$: 0.2, 2, and 100.
Calibration

- Parameter uncertainty: we have posteriors for the coefficients of the shock process, we can interpret the sample averages that were used to calculate $\alpha$ and $\beta$ as posterior means and compute posterior standard deviations.

- Treat all parameter blocks as independent, generate parameter draws, for each parameter draw simulate the DSGE model for 200 periods using
  - only neutral technology shocks $A_t$;
  - only investment-specific technology shocks $V_t$;
  - both technology shock

- Compute the ratio of the variance of hours based on actual and model generated data.
## Three Calibrations

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Calibration 1</th>
<th>90% Cred. Intv</th>
<th>Calibration 2</th>
<th>90% Cred Intv</th>
<th>Calibration 3</th>
<th>90% Cred Intv</th>
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<td>[0.005, 0.009]</td>
<td>0.007</td>
<td>[0.005, 0.009]</td>
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<tr>
<td>$\rho_V$</td>
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<td>[0.737, 0.868]</td>
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<td>[0.737, 0.865]</td>
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<td>[0.733, 0.865]</td>
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<tr>
<td>$\sigma_A$</td>
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<td>0.007</td>
<td>[0.006, 0.008]</td>
<td>0.007</td>
<td>[0.006, 0.008]</td>
</tr>
<tr>
<td>$\sigma_V$</td>
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<td>[0.003, 0.003]</td>
<td>0.003</td>
<td>[0.003, 0.003]</td>
<td>0.003</td>
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</tr>
</tbody>
</table>

Rios-Rull, Schorfheide, Grad Students: Calibration, Estimation, and Effects of Technology Shocks
Impulse Response Functions for $\nu = 0.2$ and $\nu = 100$
## Sample Variance Ratios for Hours: Model / Data

<table>
<thead>
<tr>
<th>Shock</th>
<th>Calibration 1</th>
<th>Calibration 2</th>
<th>Calibration 3</th>
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<td></td>
<td>Mean</td>
<td>90% Intv</td>
<td>Mean</td>
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<tr>
<td>$V$</td>
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<td>[0.002, 0.017]</td>
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<tr>
<td>$A, V$</td>
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<td>[0.003, 0.020]</td>
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</table>
DSGE Model Estimation

- Alternatively we can estimate the DSGE model directly.
- To make the DSGE model estimation comparable to the VAR estimation (see below) we will by using the following three series: growth rate of investment price \( \Delta \ln P^k_t \), labor productivity growth \( \Delta \ln Y_t/H_t \), and hours worked \( H_t \).
- Notice: so far we have three observables and two shocks, which means that the likelihood function is degenerate.
- To overcome this degeneracy, we introduce a preference shock, that is we let \( B_t \) evolve according to

\[
\ln(B_t/B) = \rho_b \ln(B_{t-1}/B) + \sigma_b \epsilon_{b,t}.
\]
Prior

- Details to be added...
We use MCMC methods reviewed in An and Schorfheide (2007) to obtain draws from the posterior of the DSGE model parameters. We use the Kalman smoother to obtain an estimate of total factor productivity $\ln A_t$. We compare this estimate to the estimate obtained with the “calibration” approach. Notice that we have employed different information sets.
## Calibration versus Estimation

<table>
<thead>
<tr>
<th>Name</th>
<th>Calibration 1</th>
<th>Estimation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
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<tr>
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<td>0.013</td>
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<tr>
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</tr>
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<td>γV</td>
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<td>[0.005, 0.009]</td>
</tr>
<tr>
<td>ρV</td>
<td>0.799</td>
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</tr>
<tr>
<td>ρB</td>
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<td>[0.000, 0.000]</td>
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<tr>
<td>σA</td>
<td>0.007</td>
<td>[0.006, 0.008]</td>
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<td>0.003</td>
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### Sample Variance Ratios for Hours: Model / Data

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<tr>
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<tr>
<td>$A$</td>
<td>.002</td>
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Calibration, Estimation, and Effects of Technology Shocks
Estimation: Capital Growth and Total Factor Productivity

Total Factor Productivity

Capital Stock
Finally, we will use a structural VAR to tackle our substantive question.

Let \( y_t \) be composed of the growth rates of the investment goods price and labor productivity, and the log level of hours worked.

Here is a (structural) VAR:

\[
y_t = \Phi_0 + \Phi_1 y_{t-1} + \ldots + \Phi_p y_{t-p} + \Phi \epsilon_t
\]

Our interpretation: the vector \( \epsilon_t \) is composed of the two technology shock innovations as well as the innovation to a third shock.

One can think of the third shock as preference shock, but we don’t have to take a stand. The innovations are normalized to have unit variance.
VAR Analysis

- Define reduced-form innovation $u_t = \Phi \epsilon_t$. Denote covariance matrix of $u_t$ by $\Sigma_u$.
- Write VAR in matrix form as linear regression model:

$$Y = X\Phi + U$$

where $T \times n$, $X$ is $T \times k$. 
We can link the DSGE model and the VAR by assuming that we estimate a VAR based on infinitely many observations generated from the DSGE model, conditional on structural parameters $\theta$. Let $E^D_\theta[\cdot]$ be the expectation under DSGE model and define the autocovariance matrices

$$\Gamma_{XX}(\theta) = E^D_\theta[x_t x'_t], \quad \Gamma_{XY}(\theta) = E^D_\theta[x_t y'_t].$$

Then we can define a VAR approximation of the DSGE model by population least squares:

$$\Phi^*(\theta) = \Gamma_{XX}^{-1}(\theta) \Gamma_{XY}(\theta), \quad \Sigma^*(\theta) = \Gamma_{YY}(\theta) - \Gamma_{YX}(\theta) \Gamma_{XX}^{-1}(\theta) \Gamma_{XY}(\theta).$$

(4)
A concern when estimating a DSGE model is that we are imposing invalid cross-coefficient restrictions on the data.

VARs are in general less restrictive and try to let the data speak.

To relax the cross-coefficient restrictions, we can use a prior distribution that has a lot of mass near the restrictions but does not dogmatically impose them:

\[
\Sigma | \theta \sim \mathcal{IW}(\lambda T \Sigma^*(\theta), \lambda T - k, n)
\]

\[
\Phi | \Sigma, \theta \sim \mathcal{N}\left(\Phi^*(\theta), \frac{1}{\lambda T} \left[\Sigma^{-1} \otimes \Gamma_{XX}(\theta)\right]^{-1}\right).
\]

The larger \( \lambda \), the more tightly the prior contours are concentrated.
The subspace generated by the DSGE model restrictions is shown in the figure. The prior for misspecification parameters $\Phi^\Delta$: Shape of contours determined by Kullback-Leibler distance. $\Phi^*(\theta)$: Cross-equation restriction for given value of $\theta$. Subspace generated by the DSGE model restrictions.
Identification

- To answer our substantive questions we need to identify the technology shocks, that is, we need to parameterize the VAR in terms of $\Phi_\epsilon$ instead of $\Sigma$.
- Let $\Sigma_{tr}$ be the Cholesky factor of $\Sigma$ and $\Omega$ an orthonormal matrix. Then
  \[ \Phi_\epsilon = \Sigma_{tr} \Omega \]
- Our prior for $\Sigma$ induces a prior for $\Sigma_{tr}$. We only need to add a prior for $\Omega$. 

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Calibration, Estimation, and Effects of Technology Shocks
Identification

- In the DSGE model we can calculate:

\[
\left( \frac{\partial y_t}{\partial \epsilon'_t} \right)_{DSGE} = A(\theta),
\]

say. Then use QR decomposition of \( A(\theta) \) to decompose \( A(\theta) \) into a lower triangular matrix and an orthonormal matrix \( \Omega^*(\theta) \).

- For the VAR analysis we can now use:

\[
\Phi_\epsilon = \Sigma_{tr} \Omega^*(\theta)
\]

- Hence, along the restriction function the VAR impulse responses to structural shocks will closely resemble the DSGE model impulse responses, at least in the short run.
The DSGE-VAR

- We now have the following hierarchical model:
  - Likelihood function: \( p(Y|\Phi, \Sigma) \)
  - Prior for DSGE model parameters: \( p(\theta) \)
  - Prior for VAR parameters: \( p(\Phi, \Sigma, \Omega|\theta, \lambda) \)

- Joint distribution (conditional on \( \lambda \)):

  \[
p(Y|\Phi, \Sigma)p(\Phi, \Sigma|\theta, \lambda)p(\Omega|\theta)p(\theta)
  \]

- Use MCMC methods described in Del Negro and Schorfheide (2004) to generate draws from the joint posterior distribution.
Choosing the Hyperparameter $\lambda$

- We can study the fit of the DSGE model and determine by how much the cross-coefficient restrictions need to be relaxed by examining the marginal likelihood function of the hyperparameter $\lambda$:

$$p(Y|\lambda) = \int p(Y|\Phi, \Sigma_u,)p(\Phi, \Sigma, \Omega, \theta|\lambda)d(\theta, \Phi, \Sigma, \Omega).$$ (6)

- The marginal likelihood penalizes the in-sample-fit of the estimated VAR by a measure of complexity. The larger $\lambda$, the more restricted the prior, the smaller the model complexity, and the smaller the penalty.
## DSGE-VAR Estimates

<table>
<thead>
<tr>
<th>Name</th>
<th>$\text{DSGE-VAR}(\lambda = \infty)$</th>
<th>$\text{DSGE-VAR}(\lambda = 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.353 [0.322, 0.386]</td>
<td>0.360 [0.327, 0.395]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.990</td>
<td>0.990</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.013</td>
<td>0.013</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.229 [0.056, 0.395]</td>
<td>0.484 [0.151, 0.815]</td>
</tr>
<tr>
<td>$\ln H^*$</td>
<td>-0.029 [-0.064, 0.007]</td>
<td>-0.031 [-0.070, 0.004]</td>
</tr>
<tr>
<td>$\gamma_A$</td>
<td>0.000 [-0.001, 0.001]</td>
<td>0.000 [-0.001, 0.001]</td>
</tr>
<tr>
<td>$\gamma_V$</td>
<td>0.007 [0.005, 0.008]</td>
<td>0.007 [0.005, 0.009]</td>
</tr>
<tr>
<td>$\rho_A$</td>
<td>0.000 [0.000, 0.000]</td>
<td>0.000 [0.000, 0.000]</td>
</tr>
<tr>
<td>$\rho_V$</td>
<td>0.727 [0.652, 0.800]</td>
<td>0.615 [0.506, 0.725]</td>
</tr>
<tr>
<td>$\rho_B$</td>
<td>0.970 [0.952, 0.989]</td>
<td>0.958 [0.931, 0.985]</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>0.007 [0.007, 0.008]</td>
<td>0.007 [0.006, 0.007]</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>0.003 [0.003, 0.004]</td>
<td>0.003 [0.003, 0.003]</td>
</tr>
<tr>
<td>$\sigma_B$</td>
<td>0.010 [0.008, 0.011]</td>
<td>0.008 [0.006, 0.009]</td>
</tr>
<tr>
<td>$\ln p(\mathbf{Y}</td>
<td>\lambda)$</td>
<td>2278.14</td>
</tr>
</tbody>
</table>
DSGE versus DSGE-VAR($\lambda = \infty$)
DSGE-VAR($\lambda = 1$) versus DSGE-VAR($\lambda = \infty$)
### Sample Variance Ratios for Hours: Model / Data

<table>
<thead>
<tr>
<th>Name</th>
<th>Mean</th>
<th>90% Intv</th>
<th>DSGE-VAR($\lambda = \infty$)</th>
<th>Mean</th>
<th>90% Intv</th>
<th>DSGE-VAR($\lambda = 1$)</th>
<th>Mean</th>
<th>90% Intv</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>.004</td>
<td>[.000, .008]</td>
<td>.004</td>
<td>[.000, .009]</td>
<td>.128</td>
<td>[.004, .249]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$V$</td>
<td>.008</td>
<td>[.000, .017]</td>
<td>.010</td>
<td>[.000, .021]</td>
<td>.030</td>
<td>[.001, .070]</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A, V$</td>
<td>.012</td>
<td>[.000, .025]</td>
<td>.014</td>
<td>[.000, .029]</td>
<td>.158</td>
<td>[.001, .298]</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Population Variance Decomposition

<table>
<thead>
<tr>
<th>Name</th>
<th>Mean</th>
<th>90% Intv</th>
<th>DSGE-VAR((\lambda = \infty))</th>
<th>DSGE-VAR((\lambda = 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>90% Intv</td>
<td>Mean</td>
<td>90% Intv</td>
</tr>
<tr>
<td>A</td>
<td>.004</td>
<td>[.000, .010]</td>
<td>.005</td>
<td>[.000, .011]</td>
</tr>
<tr>
<td>V</td>
<td>.011</td>
<td>[.000, .023]</td>
<td>.012</td>
<td>[.000, .024]</td>
</tr>
<tr>
<td>A, V</td>
<td>.015</td>
<td>[.000, .033]</td>
<td>.017</td>
<td>[.000, .035]</td>
</tr>
</tbody>
</table>

Rios-Rull, Schorfheide, Grad Students  Calibration, Estimation, and Effects of Technology Shocks
Deterministic Trends

- We repeat the calibration and the estimation of the DSGE model for a version of the model with deterministic trends in the two technology processes:

\[
\begin{align*}
\ln A_t - \ln A_0 - \gamma_a t &= \rho_{a,1}(\ln A_{t-1} - \ln A_0 - \gamma_a t) \\
&\quad + \rho_{a,2}(\ln A_{t-2} - \ln A_0 - \gamma_a t) + \sigma_a \epsilon_{a,t} \\
\ln V_t - \ln V_0 - \gamma_v t &= \rho_{v,1}(\ln V_{t-1} - \ln V_0 - \gamma_v t) \\
&\quad + \rho_{v,2}(\ln V_{t-2} - \ln V_0 - \gamma_v t) + \sigma_v \epsilon_{v,t}.
\end{align*}
\]
Deterministic Trends

- The point estimates for the sample 1955:I to 2006:IV are given by

\[
\begin{align*}
(\ln A_t - 4.841) &= 1.028(\ln A_{t-1} - 4.841) \\
&\quad - 0.055\rho_{a,2}(\ln A_{t-2} - 4.841) + 0.007\epsilon_{a,t} \\
(\ln V_t + 0.320 - 0.008t) &= 1.766(\ln V_{t-1} + 0.320 - 0.008t) \\
&\quad - 0.773(\ln V_{t-2} + 0.320 - 0.008t) + 0.003\epsilon_{v,t}.
\end{align*}
\]

- If the sum of the AR coefficients is 1, the model reduces to the stochastic trend specification.

- We re-parameterize the exogenous shocks in terms of partial autocorrelations: \( \rho_1 = \psi_1(1 - \psi_2); \ \rho_2 = \psi_2. \)
## Three Calibrations

<table>
<thead>
<tr>
<th></th>
<th>Calibration 1</th>
<th>Calibration 2</th>
<th>Calibration 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>90% Cred. Intv</td>
<td>Mean</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.990</td>
<td></td>
<td>0.990</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.013</td>
<td></td>
<td>0.013</td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.200</td>
<td></td>
<td>2.000</td>
</tr>
<tr>
<td>$\gamma_V$</td>
<td>0.007</td>
<td>[0.005, 0.009]</td>
<td>0.007</td>
</tr>
<tr>
<td>$\psi_{1,A}$</td>
<td>0.980</td>
<td></td>
<td>0.980</td>
</tr>
<tr>
<td>$\psi_{2,A}$</td>
<td>-0.049</td>
<td>[-0.161, 0.067]</td>
<td>-0.050</td>
</tr>
<tr>
<td>$\psi_{1,V}$</td>
<td>0.980</td>
<td></td>
<td>0.980</td>
</tr>
<tr>
<td>$\psi_{2,V}$</td>
<td>-0.770</td>
<td>[-0.832, -0.701]</td>
<td>-0.770</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>0.007</td>
<td>[0.006, 0.008]</td>
<td>0.007</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>0.003</td>
<td>[0.003, 0.003]</td>
<td>0.003</td>
</tr>
</tbody>
</table>
### Sample Variance Ratios for Hours: Model / Data

<table>
<thead>
<tr>
<th>Shock</th>
<th>Calibration 1 Mean</th>
<th>90% Intv</th>
<th>Calibration 2 Mean</th>
<th>90% Intv</th>
<th>Calibration 3 Mean</th>
<th>90% Intv</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0.003</td>
<td>[.001, 0.005]</td>
<td>0.09</td>
<td>[0.04, 0.13]</td>
<td>0.31</td>
<td>[0.14, 0.45]</td>
</tr>
<tr>
<td>V</td>
<td>0.007</td>
<td>[.003, 0.010]</td>
<td>0.22</td>
<td>[0.09, 0.34]</td>
<td>0.98</td>
<td>[0.34, 1.54]</td>
</tr>
<tr>
<td>A, V</td>
<td>0.010</td>
<td>[.005, 0.015]</td>
<td>0.31</td>
<td>[0.15, 0.46]</td>
<td>1.29</td>
<td>[0.56, 1.91]</td>
</tr>
</tbody>
</table>
Bayesian Estimation

• We now estimate the deterministic trend model using the following series: labor productivity (log level); hours worked (log level); investment-specific technology (log level)

• We also re-estimate the stochastic growth version of the DSGE model, using log levels (instead of growth rates) of labor productivity and investment-specific technology. The likelihood is constructed as in Chang, Doh, and Schorfheide (2007).

• For the log-level estimation we parameterize the DSGE model in terms of $\ln Y_0$ rather than $\ln A_0$.

• Posterior odds in favor of stochastic trend are 20 to 1.
## Posterior Estimates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Deterministic Trend Mean</th>
<th>90% Cred. Intvl</th>
<th>Stochastic Trend Mean</th>
<th>90% Cred. Intvl</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.990</td>
<td></td>
<td>0.990</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.013</td>
<td></td>
<td>0.013</td>
<td></td>
</tr>
<tr>
<td>$\nu$</td>
<td>0.670</td>
<td>[0.296, 1.038]</td>
<td>0.302</td>
<td>[0.050, 0.533]</td>
</tr>
<tr>
<td>$\gamma_A$</td>
<td>-0.001</td>
<td>[-0.002, -0.001]</td>
<td>-0.001</td>
<td>[-0.002, 0.001]</td>
</tr>
<tr>
<td>$\gamma_V$</td>
<td>0.007</td>
<td>[0.007, 0.008]</td>
<td>0.007</td>
<td>[0.005, 0.008]</td>
</tr>
<tr>
<td>$\psi_{1,A}$</td>
<td>0.975</td>
<td>[0.962, 0.990]</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>$\psi_{2,A}$</td>
<td>-0.087</td>
<td>[-0.202, 0.041]</td>
<td>0.121</td>
<td>[0.038, 0.207]</td>
</tr>
<tr>
<td>$\psi_{1,V}$</td>
<td>0.990</td>
<td>[0.988, 0.994]</td>
<td>1.000</td>
<td></td>
</tr>
<tr>
<td>$\psi_{2,V}$</td>
<td>-0.728</td>
<td>[-0.807, -0.646]</td>
<td>0.714</td>
<td>[0.636, 0.794]</td>
</tr>
<tr>
<td>$\rho_B$</td>
<td>0.970</td>
<td>[0.952, 0.990]</td>
<td>0.972</td>
<td>[0.955, 0.993]</td>
</tr>
<tr>
<td>$\sigma_A$</td>
<td>0.007</td>
<td>[0.007, 0.008]</td>
<td>0.007</td>
<td>[0.007, 0.008]</td>
</tr>
<tr>
<td>$\sigma_V$</td>
<td>0.003</td>
<td>[0.003, 0.004]</td>
<td>0.003</td>
<td>[0.003, 0.004]</td>
</tr>
<tr>
<td>$\sigma_B$</td>
<td>0.011</td>
<td>[0.010, 0.013]</td>
<td>0.010</td>
<td>[0.009, 0.011]</td>
</tr>
<tr>
<td>$\ln p(Y)$</td>
<td>2264.74</td>
<td></td>
<td>2267.60</td>
<td></td>
</tr>
</tbody>
</table>
## Sample Variance Ratios for Hours: Model / Data

<table>
<thead>
<tr>
<th>Shock</th>
<th>Deterministic Trend</th>
<th>Stochastic Trend</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>90% Intv</td>
</tr>
<tr>
<td>$A$</td>
<td>0.03</td>
<td>[0.01, 0.06]</td>
</tr>
<tr>
<td>$V$</td>
<td>0.06</td>
<td>[0.01, 0.10]</td>
</tr>
<tr>
<td>$A, V$</td>
<td>0.10</td>
<td>[0.02, 0.18]</td>
</tr>
</tbody>
</table>
Impulse Response Functions for Deterministic and Stochastic Trend Version