The Generalized Euler Equation and the Bankruptcy-Sovereign Default Problem

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Very Preliminary and Incomplete

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Motivation: Look at models where there is unilateral default

- Models of debt with unilateral default – both household debt and sovereign debt – are workhorses in the quantitative literature.
- Examples include Eaton and Gersovitz (1981), Chatterjee et al. (2007), Livshits et al. (2007), Arellano (2008)), Arellano and Ramanarayanan (2012), Arellano et al. (2019), and many others.
- These models are often solved numerically without theoretical characterizion.
- Such problem involves two different properties often confused

  1. Coarse set of financial instruments (either pay in full or pay nothing and be kicked out of financial markets)
  2. Lack of Commitment to when to do so. Instead of a maximization problem it is a game against future selves (also taking account of lenders). We are only interested in Markov Perfect Equilibria.

- We characterize the solution under commitment by functional equations that determine how much to save and when to default (and an associated value function). They include an Euler Equation (EE).
- We revisit the standard case with short term debt and no commitment (Arellano (2008)) and describe the theoretical properties as characterized by (Clausen and Strub (2020)).
- They involve an optimal saving decision, which is a Generalized Euler Equation (GEE) (or Euler Equation with derivatives of future actions). No expression for price derivatives is needed.
- We then characterize the equilibrium with long term debt, showing the properties of the price functions and giving a formulation of the Markov optimality conditions in the long-term debt case that does not rely on price derivatives.
- As set of functional equations in savings functions, default regions, and auxiliary functions (value fns and other).
- We also characterize the equilibrium in Arellano et al. (2019)

Environment: Simplest Default Model

- Endowment $\epsilon \in [\underline{\epsilon}, \overline{\epsilon}]$ is iid with cdf F and density f.
- with expected value of autarky

$$\overline{v} = \int_{\underline{\epsilon}}^{\overline{\epsilon}} u(\epsilon) + \frac{\beta}{1-\beta} E[u(c)] = \int_{\underline{\epsilon}}^{\overline{\epsilon}} u(\epsilon) + \beta \overline{v}$$

- Borrowing of uncontingent debt in competitive lending market at price $q(b')$.
- Borrowing $b' > 0$, debt pays coupon 1, each period a fraction λ of the debt matures ($\lambda = 1$ short term debt).
- Standard $u(c)$ and relative impatience, $\beta < R^{-1}$
- After default, agent reverts to financial autarky:

$$V^A(\epsilon) = u(\epsilon) + \frac{\beta}{1-\beta} E[u(c)] = u(\epsilon) + \beta \, \overline{v}$$

- Price of Debt reflects expected loses from possible default.

Nature of the Problem

- Incomplete market problem:

  1. Only two states pay or not pay
  2. Quantity: if not pay (default) restricted to zero payments
  3. If not paying exclusion forever.

- But also the timing of circumstances under which commit:

  1. Ex-ante **Commitment**
  2. Ex-post **Not Commitment**

The Problem With Commitment

What does it mean to have commitment?

- To choose ex-ante when to default: *i.e.* to commit to when to default
- So no choice of default within the period.
- With Commitment Long and Short Term is the Same.
  - Proof. Given one, build the other.
- Two alternative recursive timings
  1. Choose today when to default tomorrow
  2. Choose circumstances of when to default before realization of shock but commitment to expected value of debt

Timing 1: when ϵ is such that the agent is committed to pay

- Let $$\Psi(b, \epsilon^c, \epsilon) = \begin{cases} v^A(\epsilon), & \epsilon < \epsilon^c, \\ \Omega(b, \epsilon), & \epsilon \geq \epsilon^c. \end{cases}$$
- where $\Omega(b, \epsilon)$ for $\epsilon \geq \epsilon^c$

$$\Omega(b, \epsilon) = \max_{b', \epsilon'^c} \left\{ u(c) + \beta \int \Psi(b', \epsilon'^c, \epsilon') f(d\epsilon') \right\}$$

$$= \max_{b', \epsilon'^c} \left\{ u(c) + \beta \int^{\epsilon'^c} v^A(\epsilon') f(d\epsilon') + \beta \int_{\epsilon'^c} \Omega(b', \epsilon') f(d\epsilon') \right\}$$

s.t. $$c + b = b' \; \frac{[1 - F(\epsilon'^c)]}{1 + r} + \epsilon$$

- Substituting in the constraint for $\epsilon \geq, \epsilon^c$ yields

$$\Omega(b, \epsilon) = \max_{b', \epsilon'^c} \; u\left(b' \; \frac{[1 - F(\epsilon'^c)]}{1 + r} + \epsilon - b \right) +$$

$$\beta \int_{\underline{\epsilon}}^{\epsilon'^c} \left(u(\epsilon) + \beta \overline{v} \right) \; f(d\epsilon') + \beta \int_{\epsilon'^c}^{\overline{\epsilon}} \; \Omega(b', \epsilon'^c, \epsilon') \; f(d\epsilon')$$

Timing 1: FOC & Envelope

$$b'(\epsilon): \qquad \frac{[1-F(\epsilon'^c)]}{1+r}\ u_c\left(b'\ \frac{[1-F(\epsilon'^c)]}{1+r}+\epsilon-b\right)=-\beta\int_{\epsilon'^c}\Omega_b(b',\epsilon',\epsilon'^c)\ f(d\epsilon')$$

$$\epsilon'^c: \qquad \frac{-f(\epsilon^c)\ b'}{1+r}\ u_c\left(b'\ \frac{[1-F(\epsilon^c)]}{1+r}+\epsilon-b\right)=\beta\ f(\epsilon^c)\ \left[\Omega(b',\epsilon'^c,\epsilon'^c)-u(\epsilon'^c)\right.$$

$$\text{Env } b: \qquad \Omega_b(b,\epsilon,\epsilon^c)=-u_c\left(b'\ \frac{[1-F(\epsilon'^c)]}{1+r}+\epsilon-b\right),$$

$$\frac{[1-F(\epsilon'^c)]}{1+r}\ u_c = \beta \int_{\epsilon'^c} u'_c\ dF(\epsilon)',$$

$$\Omega(b,\epsilon^c) = u(\epsilon^c) + \beta\overline{v} - \frac{b}{[1-F(\epsilon^c)]}\ \int_{\epsilon^c} u_c\ f(d\epsilon),$$

$$\Omega(b,\epsilon) = \max_{b',\epsilon'^c}\ u\left(b'\ \frac{[1-F(\epsilon'^c)]}{1+r} + \epsilon - b\right) +$$

$$\beta \int_{\underline{\epsilon}}^{\epsilon'^c} \left(u(\epsilon) + \beta\overline{v}\right)\ f(d\epsilon') + \beta \int_{\epsilon'^c}^{\overline{\epsilon}}\ \Omega(b',\epsilon'^c,\epsilon')\ f(d\epsilon').$$

which implies discontinuity (and regret) of $\Psi(b,\epsilon^c,\epsilon)$ at $\epsilon = \epsilon^c$.

Timing 2: Solving the problem at night: Commited to pay a in Expectations

- $$\Phi(a) = \max_{m,\epsilon^c,c(\epsilon),a'(\epsilon)} \left\{ \int_{\underline{\epsilon}}^{\epsilon^c} \left(u(\epsilon) + \beta \overline{v} \right) \; f(d\epsilon) + \quad \int_{\epsilon^c} u[c(\epsilon)] \; f(d\epsilon) + \beta \int_{\epsilon^c} \Phi[a'(\epsilon)] \; f(d\epsilon) \right\} \qquad \textbf{s.t.}$$

$$a = [1 - F(\epsilon^c)] \; m$$

$$c(\epsilon) = \epsilon + \frac{a'(\epsilon)}{1+r} - m, \qquad \text{when } \epsilon > \epsilon^c, \qquad\qquad \text{price of debt is } \frac{1}{1+r}.$$

- Substituting in the constraints yields

$$\Phi(a) = \max_{\epsilon^c, a'(\epsilon)} \left\{ \int_0^{\epsilon^c} \left(u(\epsilon) + \beta \overline{v} \right) f(d\epsilon) + \int_{\epsilon^c} u \left[\epsilon + \frac{a'(\epsilon)}{1+r} - \frac{a}{1 - F(\epsilon^c)} \right] f(d\epsilon) + \beta \int_{\epsilon^c} \Phi[a'(\epsilon)] f(d\epsilon) \right\}$$

- The FOC wrt $a'(\epsilon)$ and ϵ^c and the envelope are

$$u_c \left[\epsilon + \frac{a'(\epsilon)}{1+r} - \frac{a}{1 - F(\epsilon^c)} \right] = -\beta \; (1+r) \; \Phi_a[a'(\epsilon)], \qquad \text{for } \epsilon > \epsilon^c$$

$$u(\epsilon^c) + \beta \overline{v} = u[c(\epsilon^c)] + \beta \Phi[a'(\epsilon^c)] + \frac{a}{[1 - F(\epsilon^c)]^2} \int_{\epsilon^c} u_c[c(\epsilon)] dF) = u + \beta \Phi' + \frac{m}{1 - F(\epsilon^c)} \int_{\epsilon^c} u_c dF$$

$$\Phi_a(a) = -\frac{1}{1 - F(\epsilon^c)} \int_{\epsilon^c} u_c[c(\epsilon)] f(d\epsilon)$$

Timing 2: More Algebra

- Forwarding the envelope condition yields

$$u_c \left[\epsilon + \frac{a'(\epsilon)}{1+r} - \frac{a}{1-F(\epsilon^c)}\right] = \beta\ (1+r)\ \frac{1}{1-F[\epsilon'^c(a'(a,\epsilon))]}\ \int_{\epsilon'^c} u_c[c(a'(a,\epsilon),\epsilon')]f(d\epsilon'),$$

$$u(\epsilon^c) + \beta\overline{v} = u[c(\epsilon^c)] + \beta\Phi[a'(\epsilon^c)] + \frac{m}{1-F(\epsilon^c)}\int_{\epsilon^c} u_c\ f(d\epsilon)$$

- Writing them in full glory

$$u_c \left[\epsilon + \frac{a'(a,\epsilon)}{1+r} - \frac{a}{1-F[\epsilon^c(a)]}\right]\left(1 - F(\epsilon^c[a'(a,\epsilon)])\right) =$$

$$\beta(1+r)\int_{\epsilon^c[a'(a,\epsilon)]} u_c \left[\epsilon' + \frac{a''[a'(a,\epsilon),\epsilon']}{1+r} - \frac{h(a,\epsilon)}{1-F(d[h(a,\epsilon)])}\right] f(d\epsilon')$$

$$u[\epsilon^c(a)] + \beta\overline{v} = u\left[\epsilon^c(a) + \frac{a'(a,\epsilon^c)}{1+r} - \frac{a}{1-F[\epsilon^c(a)]}\right] + \beta\Phi[a'(a,\epsilon)]$$

$$+ \frac{m}{1-F[\epsilon^c(a)]}\int_{\epsilon^c} u_c\left[\epsilon + \frac{a'(a,\epsilon)}{1+r} - \frac{a}{1-F[\epsilon^c(a)]}\right] f(d\epsilon)$$

- And compactly

$$u_c = \beta\ (1+r)\ \frac{1}{1-F[\epsilon'^c]}\ \int_{\epsilon'^c} u'_c f(d\epsilon'), \qquad \epsilon \geq \epsilon^c$$

$$u(\epsilon^c) + \beta\overline{v} = u[c(\epsilon^c)] + \beta\Phi[a'(\epsilon^c)] + \frac{m}{1-F(\epsilon^c}\ \int_{\epsilon^c} u_c[c(\epsilon)]\ \ f(d\epsilon)$$

Need to verify that they are the same

- The Euler equation is the same

$$\frac{[1 - F(\epsilon'^c)]}{1 + r} \; u_c = \beta \int_{\epsilon'^c} u'_c \; dF(\epsilon)',$$

- To verify the second condition note that for any pair $\{b, \epsilon^c\}$, the amount commited to pay in timing 2 is $a = b \; [1 - F(\epsilon^c)]$, so the relation between the two functions is

$$\Phi\left(b \; \left[1 - F\left(\epsilon^c\right)\right]\right) = \int \Psi(b, \epsilon^c, \epsilon) \; dF,$$

- which implies that the second condition is also the same:

$$u[\epsilon^c(a)] + \beta\overline{v} = u\left[\epsilon^c(a) + \frac{a'(a, \epsilon^c)}{1 + r} - \frac{a}{1 - F[\epsilon^c(a)]}\right] + \beta\Phi[a'(a, \epsilon)]$$

$$+ \frac{b}{1 - F[\epsilon^c(a)]} \int_{\epsilon^c} u_c \left[\epsilon + \frac{a'(a, \epsilon)}{1 + r} - b\right] f(d\epsilon).$$

The Problem Without Commitment: Short Term Debt

Short-Term Debt: Clausen and Strub (2020)

- Value of honoring debt, *Repaying*

$$V^R(\epsilon, b) = \max_{b'} \left\{ u\big[\epsilon - b + q(b')b'\big] + \beta \int_{\underline{\epsilon}}^{\bar{\epsilon}} \max\left\{ V^R(\epsilon', b'), V^A(\epsilon') \right\} dF \right\}$$

- Sovereign takes as given default threshold and prices $q(b')$ Ex-post Choice

$$d(b') = \min\left\{ \{\epsilon' : V^R(\epsilon', b') \geq V^A(\epsilon')\} \cup \{\bar{\epsilon'}\} \right\}$$

- Value of honoring debt becomes

$$V^R(\epsilon, b) = \max_{b'} \left\{ u\big[\epsilon - b + q(b')b'\big] + \beta \underbrace{\int_{d(b')}^{\bar{\epsilon}} \left\{ V^R(\epsilon', b') - V^A(\epsilon') \right\} dF}_{\text{value of access to credit markets}} + \beta \bar{v} \right\}$$

- So the sovereign chooses $\max\left\{ V^R(\epsilon, b), V^A(\epsilon) \right\}$

Short-Term Debt: First Order Condition

$$u_c(c)\underbrace{\left[q(b') + q_b(b')\ b'\right]}_{\text{marginal revenue}} = \beta \int_{d(b')}^{\bar{\epsilon}} u_c(c')\ dF$$

- We still need to know more about $q(b')$:
  - Where does it come from
  - Is it differentiable
  - What if it is not differentiable?
- The presence of $q_b(b')$ makes the Euler equation a Generalized Euler Equation (GEE) typical of environments with Time Inconsistency (Krusell and Smith (2003), Krusell et al. (2002), Krusell et al. (2010), Klein et al. (2008))

Short-Term Debt: Bond Price and FOC

- Bond Price

$$\frac{q(b')}{1+r} = \begin{cases} [1 - F(d(b))], & b^* < b', \\ 1, & b' \leq b^*. \end{cases}$$

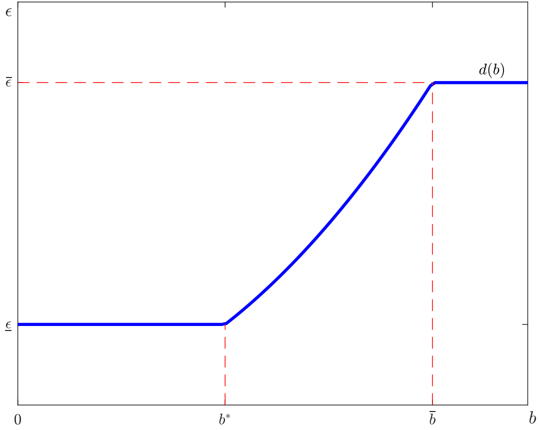
- Whether it is diffentiable is inherited from $d(b)$ (it is for $b' \neq b^*$)

$$\frac{q_b(b')}{1+r} = -f[d(b')]\ d_b(b')$$

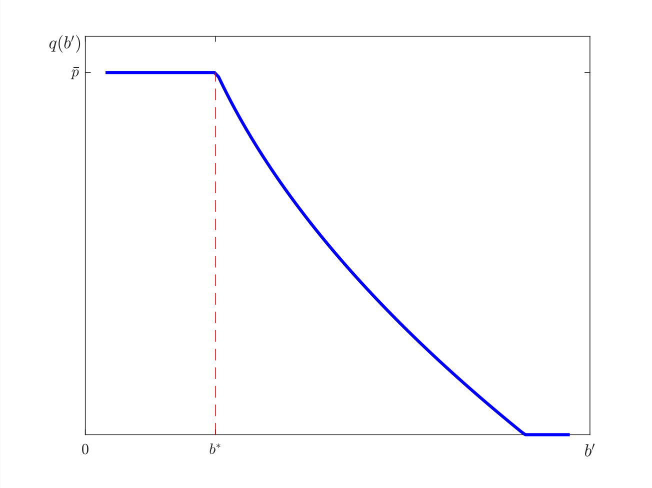
- Marginal revenue of borrowing at b'

$$q(b') + q_b(b')b' = (1+r)\{[1 - F(d(b))] - f[d(b')]\ d_b(b')\ b'\}$$

Default Threshold



- For debt $b > b^*$ there is default risk.
- $d(b)$ not differentiable at b^*. $\partial^+ d(b) > 0$, but $\partial^- d(b) = 0$.
- No analytical solution for b^*, but we know it solves $V^R(\underline{\epsilon}, b^*) = V^A(\underline{\epsilon})$.



- The kink in q at b^* makes b^* more attractive.
- Agents will choose to state at b^* to avoid lowering the price of their debt.

Short-Term Debt: GEE

- From Clausen and Strub (2020) we know that either

1. The probability of default is zero and $b' < b^*$ solves Euler Equation $(q_b(b') = 0)$

$$u_c(c) = \beta(1+r)\int u_c(c')\ dF$$

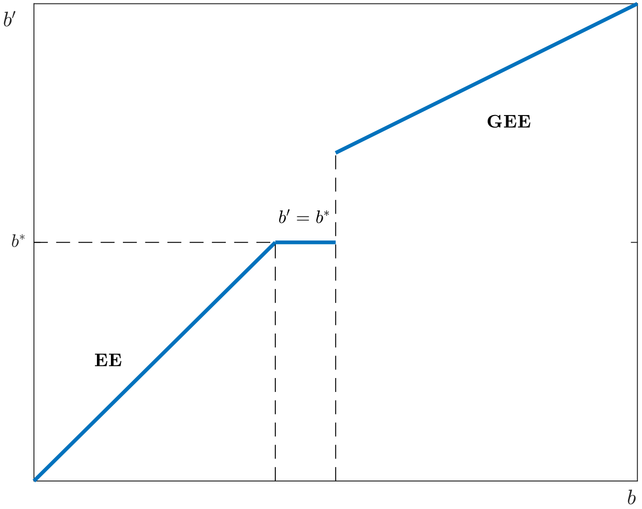
2. There is a corner and the FOC is not satisfied $(b' = b^*)$

3. The probability of default is positive, $(b' > b^*)$ and solves the GEE:

$$u_c(c)[(1 - F(d(b))) - f(d(b'))d_b(b')b'] = \beta R \int_{d(b')}^{\bar{\epsilon}} u_c(c')dF$$

- Notice that we have eliminated the price function from these functional equations

Short-Term Debt: Borrowing Policy for some (low) ϵ



- Agents stay at the risk-free limit b^* to avoid lowering price of debt
- There is a discontinuity because q_b is not differentiable at b^* and it jumps.

- With

$$\frac{[1 - F(\epsilon'^c)]}{1+r} \; u_c = \beta \int_{\epsilon'^c} u'_c \; dF(\epsilon)',$$

$$\Omega(\epsilon^c, b) = u(\epsilon^c) + \beta\overline{v} - \frac{b}{[1 - F(\epsilon^c)]} \int_{\epsilon^c} u_c \; f(d\epsilon),$$

- Without

$$\frac{[(1 - F[d(b)]) - f[d(b')] \; d_b(b') \; b']}{1+r} \; u_c = \beta \int_{d(b')}^{\bar{\epsilon}} u_c(c')dF$$

$$V^R(\epsilon, b) = u\big[d(b)\big] + \beta\overline{v}$$

The Problem With Commitment:
Long Term Debt

A Trickier Problem

Long-Term Debt: What's Different? Dilution

- The Budget Constraint with long maturity bonds (exponential decay)

$$c = \epsilon - b + q(b') \; [b' - (1-\lambda)b]$$

- The sovereign's choice of borrowing determines the value of outstanding debt

$$q(b')(1-\lambda) \; b$$

- The lack of Commitment is now double:

  1. To when to Default
  2. To how much to borrow in the future: Dilution of existing debt

- This is a harder problem to characterize while getting rid of the price.

Long-Term Debt: Sovereign's Problem

- The value of repaying debt

$$V^R(\epsilon, b) = \max_{b'} \left\{ u\left(\epsilon - b + q(b')\left[b' - (1-\lambda)b\right]\right) + \beta \int_{d(b')}^{\bar{\epsilon}} \left\{ V^R(\epsilon', b') - V^A(\epsilon') \right\} dF + \beta \bar{v} \right\}$$

$$= \max_{b'} \left\{ u\left(\epsilon - b + q(b')\left[b' - (1-\lambda)b\right]\right) + \beta W(b') \right\}$$

- When it holds, the GEE looks like

$$u_c(\cdot)\left[q(b') + q_b(b')[b' - (1-\lambda)b]\right] = -\beta W_b(b')$$

- Depends on derivative of two objects $q_b(b')$ and $W_b(b')$

Lemma. $W(b')$ is differentiable everywhere in b'.

$$W(b') = \int_{d(b')}^{\bar{\epsilon}} \{V^R(\epsilon', b') - V^A(\epsilon')\} dF + \beta \bar{v}$$

$$W_b(b') = -\int_{d(b')}^{\bar{\epsilon}} u_c(c') \left[1 + (1-\lambda) q(b'')\right] dF$$

- The marginal cost of an additional unit of borrowing is the expected marginal utility loss of paying the coupon and rolling over unmatured debt at tomorrow's price in repayment states.

- The bond price equals discounted expected payoff of lending b'.

$$\frac{q(b')}{1+r} = \int_{\underline{\epsilon}}^{\bar{\epsilon}} 1_{\{V^R(\epsilon',b') \geq V^A(\epsilon')\}} \left[1 + (1-\lambda) q(h(\epsilon', b'))\right] \, dF$$

$$= \left[1 - F(d(b'))\right] + (1-\lambda) \int_{d(b')}^{\bar{\epsilon}} q\big(h(\epsilon', b')\big) \, dF$$

- Price depends on both default $d(b')$ and future borrowing $h(b', \epsilon')$
- Changes in the price due to $d(b')$ reflect *default risk*, those due to $h(\epsilon', b')$ reflect *dilution risk*.
- Intuitively, more borrowing b' today increases borrowing tomorrow $h(\epsilon', b')$

- An operator $H(q)$ on arbtitrary prices can be defined as the updated price given an optimal response to the original

$$H(q)(b') = \bar{p}\big[1 - F(d(b';q))\big] + \bar{p}(1-\lambda)\int_{d(b';q)}^{\bar{\epsilon}} q\big(h(\epsilon', b'; q)\big)dF$$

- Note that $d(\cdot;q)$ and $h(\cdot;q)$ being implicit functions of q.

- Chatterjee and Eyigungor (2012) show existence of a fixed point $q^* h(q^*)$ that is decreasing in b' (for discrete domain of b and ϵ and using lotteries to get continuity).

- We want to strengthen what we can say about $q(b')$, since the price derivative $q_b(b')$ affects the marginal incentive to borrow.

Long-Term Debt: Bond Price

- We impose a restriction on $q(b')$ that it be the limit of a finite horizon model as $T \to \infty$. To eliminate equilibria like in Krusell and Smith (2003).

- Specifically, we consider the price of debt in the first period of a finite horizon model $q_1(b'; T)$ as T becomes large

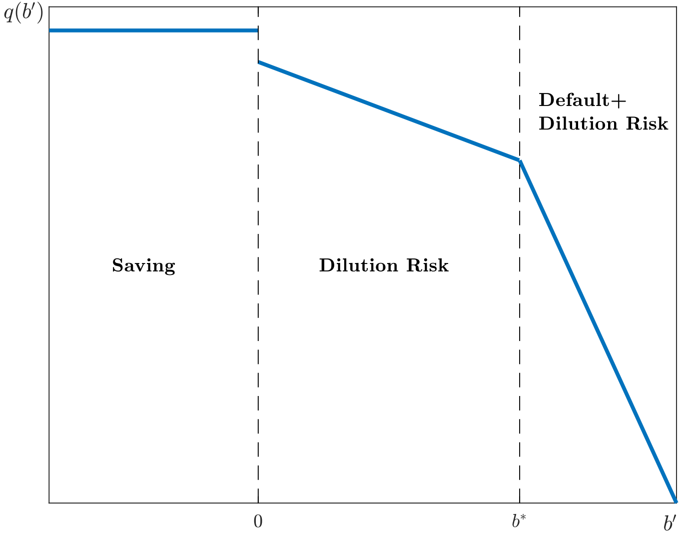
- We use backwards induction starting at $q_T(b'; T) = 0$ to get $q_{T-1}(b'; T) = \bar{p}1_{\{b'<0\}}$, ..., until $q_1(b'; T)$.

- **Bond Price**

$$q(b') = \begin{cases} \frac{1}{r+\lambda}, & b' \leq 0 \\ \frac{1}{1+r} + \frac{1+r}{1-\lambda} \int_{\underline{\epsilon}}^{\overline{\epsilon}} q[h(\epsilon', b')] \, dF, & 0 < b' \leq b^* \\ \frac{1}{1+r}[1 - F(d(b'))] + \frac{1+r}{1-\lambda} \int_{d(b')}^{\overline{\epsilon}} q[h(\epsilon', b')] \, dF, & b' > b^* \end{cases}$$

- Debt will be honored next period with certainty, but is discounted for dilution risk. Discontinuity. if there is probability of $b' > b^*$ at some point (after a sequence of bad shocks), the price today reflects this risk.
- There may be more discontinuities in the dilution zone: as many as the number of periods it takes to default with possitive probability starting with zero debt.
- Each discontinuity point indicates one less period until default is possible

Long-Term Debt: if the Limit when $T \to \infty$ exists then the Bond Price



- For no additional discontinuities in the dilution zone

Derivative for $b' \notin \{0, b^*\}$

$$|{!}q_b(b') = \underbrace{\frac{1-\lambda}{1+r}\int_{d(b')}^{\bar{\epsilon}} q_b(h(\cdot))\ h_b(\cdot)\ dF}_{\textbf{Dilution},\ b'>0} - \overbrace{\overbrace{\left[\frac{1}{1+r}1 + \frac{1-\lambda}{1+r}q(h(d(b'), b'))\right]}^{\textbf{Value of loss}} \overbrace{f(d(b'))d_b(b')}^{\textbf{Marginal P(default)}}}^{\textbf{Default},\ b'>b^*}$$

1. Borrowing $b' > b^*$ has both *default* and *dilution* terms
2. Borrowing $0 < b' < b^*$ has *dilution* risk only
3. Saving $b < 0$ has neither

Long-Term Debt: Bond Price (with T discontinuities in the dilution zone)

- Is this dilution term well-defined?

$$\int_{d(b')}^{\bar{\epsilon}} q_b[h(\cdot)] \; h_b(\cdot) dF$$

- Yes: There are three types of points $\epsilon \in [d(b'), \bar{\epsilon}]$.

1. Points s.t. $b' \notin \{0, b_1, \cdots, b_{T-1}, b^*\}$, and h_b, $q_b(h)$ are defined.
2. Points s.t. $b' \in \{0, b_1, \cdots, b_{T-1}, b^*\}$, and $h_b = 0$, $\Rightarrow q_b(h) \; h_b = 0$.
3. The remaining points where $b' \in \{0, b_1, \cdots, b_{T-1}, b^*\}$, and h_b, hence the integrand $q_b(h)h_b$, is not well-defined. But the set of those points has zero measure.

- So the GEE is well defined for all points except $\{0, b_1, \cdots, b_{T-1}, b^*\}$

Eliminating $q_b(b')$

- Use value of q_b implied by the GEE, call it $B(h, d, q)$ outside of $\{0, b_1, \cdots, b_T, b^*\}$

$$q_b = B(h, d', q) = \frac{\int_{d'} u_c[1 + (1-\lambda)q']\ dF - u_c(c)q}{u_c[h - (1-\lambda)b]}$$

- Substitute this into the expression for the bond price derivative

$$\frac{q_b}{1+r} = (1-\lambda)\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')\ h_b\ dF - [1 + (1-\lambda)\ \tilde{q}]\ \ f(d)\ d_b$$

- Substitute back into GEE

$$u_c(c)\left[q(b') + \left\{\frac{1-\lambda}{1+r}\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')h_b dF - \frac{1}{1+r}\left[1 + (1-\lambda)\tilde{q}\right] f(d) d_b\right\}[b' - (1-\lambda)b]\right]$$
$$= \beta\int_{d(b')}^{\bar{\epsilon}} u_c(c')\left[1 + (1-\lambda)q(b'')\right] dF$$

- This GEE only includes decision rules and a price function that if it exists is the limit of objects that depend only on decision rules.

- It is only satisfied when q_b is diffentiable

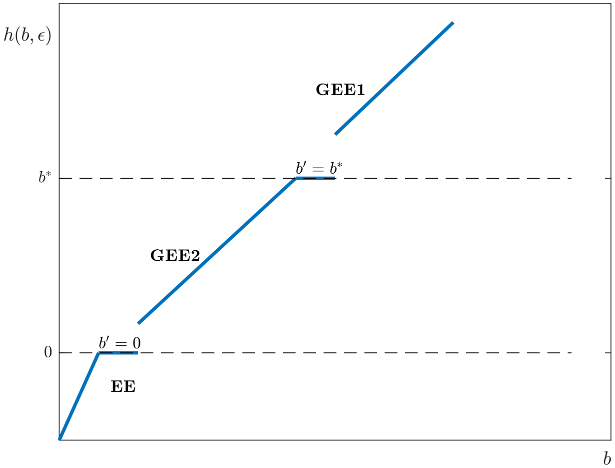
$$u_c(c)\Bigg[\overbrace{q(b')}^{\text{consumption gain from marginal borrowing}} + \underbrace{\left\{\frac{1-\lambda}{1+r}\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')\ h_b\ dF\right\}}_{\text{dilution, } b'>0}[b' - (1-\lambda)\ b] - \underbrace{\left\{\frac{1}{1+r}\left[1+(1-\lambda)\ \tilde{q}\right]\ f(d)\ d_b\right\}}_{\text{default, } b'>b^*}[b' - (1-\lambda)b]\Bigg]$$

$$= \beta \int_{d(b')}^{\bar{\epsilon}} u_c(c')\left[1+(1-\lambda)\ q(b'')\right] dF$$

Two borrowing regions that reflect different risks to creditors:

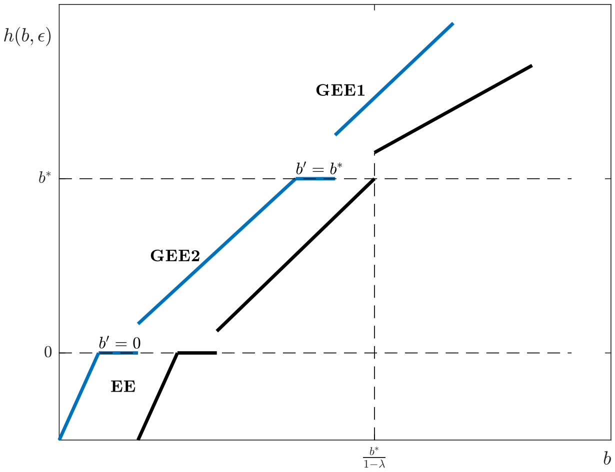
1. $b' > b^*$ the GEE reflects both default and dilution risk (GEE 1)
2. $0 < b^{'} < b^*$ the GEE reflects only dilution risk (GEE 2)

Borrowing Policy: Takes Various Forms depending on net borrowing: When Dilution is positive $b < b^*/(1-\lambda)$



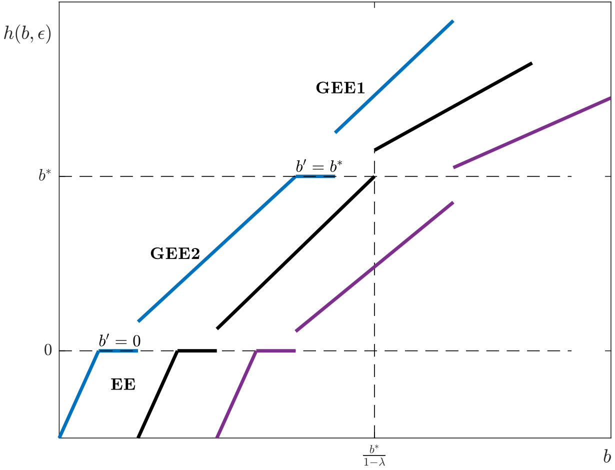
- Agents wait to borrow, due to dilution lowering the price of borrowing.
- As with short-term debt, agents stay at risky borrowing limit b^*.
- There is a discontinuity at b^*, due to the kink in the pricing function.

Borrowing Policy: When Dilution is positive $b < b^*(/1-\lambda)$ & when is zero



- Notice the location of the no staying at b^*: when there is no net borrowing.
- Note also that the discontinuities are smaller: it is because less is borrowed because the price is higher

Borrowing Policy: When dilution effect is negative ($b > \frac{b^*}{1-\lambda}$)



- There is a jump direct to higher debt without going through b^*
- As with short-term debt, agents stay at the risky borrowing limit b^* provided the dilution effect is large enough.

A Characterization of the Default Threshold

We can take a closer look at the derivative of the default threshold

$$d_b(b') = \frac{u_c\big(c(d(b'), b')\big)[1 + (1-\lambda)q(b'')]}{u_c\big(c(d(b'), b')) - u_c\big(d(b')\big)} > 1$$

- Numerator is marginal utility loss from additional debt after repayment.

- Denominator cost, in terms of marginal utility, to maintain access to financial markets.

Long-Term Debt: Summary 1:

- The value b^* is crucial (either maximum level of debt and no default or default threshold that informs all the behavior of prices and debt choices.
- Prices and debt functions exhibit jumps in various places.
- Not good to approximate with continuous functions (Arellano et al. (2016))
- We can say many things about the nature of equilibrium when default.
  - Trouble starts at zero
  - There exists a dilution zone (or many)
  - There exists a default zone
  - Dilution works to hurt lenders
  - But also the sovereign
  - For areas where the GEE is satisfied, we have a clear picture of what are the forces at work without using price derivatives
  - The multiplicity described by Aguiar and Amador (2018) may be due to the equilibrium not being the limit of finite economies.

We can describe equilibrium as set of functional aligns in h and d

- Auxiliary Functions

$$q(h(\epsilon, b)) = \frac{1}{1+r}\left\{[1 - F(d)] + (1-\lambda)\int_d q(h(h))dF\right\}$$

$$B(\epsilon, b; h, d, q) = \frac{\int_{d'} u_c[1 + (1-\lambda)q']dF - u_c q}{u_c[h - (1-\lambda)b]}$$

$$V^R(\epsilon, b) = u(\epsilon - bq[h - (1-\lambda)b) + \int_d V^R - V^A dF + \beta\bar{v}$$

- Equilibrium functional aligns

$$u_c(c)\left[q(b') + \left\{\frac{1-\lambda}{1+r}\int_{d(b')}^{\bar{\epsilon}} B(h', d'', q')h_b dF - \frac{1}{1+r}[1 + (1-\lambda)\tilde{q}]\, f(d)d_b\right\}[b' - (1-\lambda)b]\right]$$

$$= \beta\int_{d(b')}^{\bar{\epsilon}} u_c(c')\left[1 + (1-\lambda)q(b'')\right]dF$$

$$V^R(d, \epsilon) = V^A(d), \qquad V^R(\underline{\epsilon}, b^*) = V^A(\underline{\epsilon})$$

- The most common way to solve these models is value function iteration on a discrete grid. Very slow. Need to iterate between $V(\epsilon, b; q)$ and q.
- Arellano et al. (2016) use euler equation to solve short-term debt problem numerically, but assume the GEE always holds.
- Hatchondo et al. (2010) compare various VFI algorithms to solve the short-term debt problem, but assess their accuracy using Euler residuals.
- Our characterization suggests using discretization and limit the GEE and auxiliary functions to charaterize only interior solutions.

The Partial Default Model Arellano et al. (2019)

- This is a model of partial default.

- What is not paid accumulates at rate $\overline{R}$, and reduces output tomorrow. Think of voluntary and involuntary borrowing from the point of view of the lenders.

- Endowment ϵ with density f and cdf F. Asset position is A, more precisely $A > 0$ is the amount to pay today. Unpaid debt is $0 \leq D \leq A$, it accumulates at exogenous rate $\overline{R}$ and it reduces the endowment tomorrow a fraction $[1 - \psi(D)]$. New emissions of (voluntary) debt are B, become part of A' one for one, and are priced at $Q(A, B, D)$.

- The sovereign solves

$$V(a,y,z) = \max_{b,d,c} \left\{ u(c) + \beta \sum_{z'} \pi(z',z)\ V(a',y',z') \right\}, \qquad \text{s.t.}$$

$$c = y - (1-d)\ a + q(a',d,z)\ b,$$

$$a' = \delta a + (R-\delta)\ \kappa\ d\ a + b,$$

- Value of Debt

$$H(a,y,z) = [1 - d(a,y,z)] + \frac{1}{R}\ [\delta + (R-\delta)\ \kappa\ d(a,y,z)] \times$$
$$\sum_{z'} \pi(z',z)\ H\left[a'(a,y,z), z'\Psi[d(a,y,z),z'], z'\right].$$

- Eq requires that borrowing $b(a,y,z)$, partial default $d(a,y,z)$, the value of debt $H(a,y,z)$ and the bond price function $q(a',d,z)$ solve sovereign's stisfy the value of debts and
  - Bond prices $q(a',d,z)$ yield expected zero profits to lenders so that

$$q(a',d,z) = \frac{1}{R}\ \sum_{z'} \pi(z',z)\ H\left(a', z'\Psi(d,z'), z'\right).$$

To derive the GEE, first obtain FOC's using prices

- let the debt burden be

$$\Lambda(d', q') \equiv \underbrace{1 - d'}_{\textbf{repayment}} + \underbrace{[\delta + (R-\delta)\ \kappa\ d']\ q'}_{\textbf{further repayments at price } q'} .$$

- Substituting in the price function

$$q(a', d, z) = \frac{1}{R}\ E\left\{\left[1 - d'\right] + \left[\delta + (R-\delta)\ \kappa\ d'\right]\ q'\right\} = \frac{1}{R}\ E\left\{\Lambda(d', q')\right\},$$

- The FOC and envelope become as function of prices and debt burden

$$u_c\ [\underbrace{q + q_{a'}\ b}_{\textbf{borrowing gain}}] = \beta\ E\{u_c'\ \underbrace{\Lambda'}_{\textbf{debt burden}}\},$$

$$u_c\ \underbrace{[a + (q_{a'}\ (R-\delta)\ \kappa\ a + q_d)\ b]}_{\textbf{partial default gain}} = \beta\ E\{u_c'\ [\underbrace{(R-\delta)\ \kappa\ a\ \Lambda'}_{\textbf{debt burden from defaulted coupons}} - \underbrace{z'\ \Psi_d}_{\textbf{default cost}}]\}.$$

Define auxiliary functions that do not have future derivatives

- Price derivatives

$$q_{a'} = \frac{1}{R}\ E\left\{\Lambda'_d\ d'_a + \Lambda'_q\left[q'_{a'}\left[\delta + b'_a + (R-\delta)\ \kappa\ \left(a'\ d'_a + d'\right)\right] + q'_d\ d'_a\right]\right\}$$

$$q_d = \frac{1}{R}\ E\left\{\Lambda'_d\ d'_y\ \Psi'_d\ z' + \Lambda'_q\left[q'_{a'}\ \left[b'_y + (R-\delta)\ \kappa\ a'\ d'_y\right] + q'_d\ d'_y\right]\Psi'_d\ z'\right\}.$$

- Use FOC to define functions $\mathcal{B}(a,y,z)$ and $\mathcal{D}(a,y,z)$ depend only on decision rules and the current price.

$$q_{a'} = \frac{\beta\ E\{u'_c\ \Lambda' - q\ u_c\}}{u_c} \equiv \mathcal{B}(a,y,z),$$

$$q_{a'}(R-\delta)\kappa a + q_d = \frac{\beta\ E\{u_c\ [\ (R-\delta)\ \kappa\ a\ \Lambda'\ - z'\ \Psi'_d]\} - a\ u_c}{b\ u_c} \equiv \mathcal{D}(a,y,z).$$

Obtain expressions for future derivatives

- The derivative with respect to debt due a' is

$$q_{a'} = \frac{1}{R} E \Big\{ \Lambda_d' \; d_a'$$

direct loss from not paying

$$+ \Lambda_q' \times$$

continuation amount gets diluted because of the

$$\left[\mathcal{B}'(\delta + b_a' + (R - \delta)\, \kappa \, d') + \mathcal{D}' \; d_a' \right] \Big\}$$

change in future prices with more debt.

- The derivative with respect to partial default d is

$$q_d = \frac{1}{R} E \Big\{ \Lambda_d' \; d_y' \; \Psi_d' \; z'$$

lower output tomorrow yields more default

$$+ \Lambda_q' \times$$

continuation amount gets diluted because of the

$$\left[(\mathcal{B}' b_y' + \mathcal{D}' \; d_y') \; \Psi_d' \; z' \right] \Big\}$$

change in future prices with lower output.

- Substituting Back in the FOC yiedls the GEE

Conclusion

- We have characterized the equilibrium of unilateral default problem with commitment. It is very different than the standard problem with short term debt that has no commitment to the circumstances of default.
- We characterized the equilibrium of unilateral default problem without commitment for long term debt and to partial default.
- For interior solutions, the GEE describes the optimal borrowing policy.
- The GEE fails to capture tradeoffs at choices where the price is not differentiable, which has many points of this type, but we can still describe the optimal policy.
- Thank you!

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