The Generalized Euler Equation and the Bankruptcy–Sovereign Default Problem

Xavier Mateos-Planas and José-Víctor Ríos-Rull

Univ of Minnesota, UCL, Mpls Fed, Queen Mary University of London

Very Preliminary

February 11, 2015

Introduction: The problem

- The unilateral default problem and its associated dilution of long term debt problem has long been a problem of interest

  (Eaton and Gersovitz (1981), Chatterjee, Corbae, Nakajima, and Ríos-Rull (2007), Livshits, MacGee, and Tertilt (2007), Arellano (2008), Arellano and Ramanarayanan (2012), Arellano, Mateos-Planas, and Ríos-Rull (2013), Adam and Grill (2012), Passadore and Xandri (2015), Perez (2015) and many others).

- This problem is typically characterized either numerically or equilibria is constructed to have some properties (via 2 extreme possible values for the shock for instance).

Main features of this environment

1. Borrowers have no commitment to return a loan. They sometimes default in circumstances that are different to those that they would have liked to have committed to.

2. If long term debt exists, the borrower cannot commit to limit additional borrowing in the future and there are no well defined seniority rules for debt.

3. There are multiple lenders and new lenders are always available. Past lenders cannot limit the activities of future lenders, at least in the absence of default.

4. Some form of punishment follows default. Typically, it is either output (or utility) reduction, or limited access to future borrowing, or both.

What we do

- We provide the characterization of (Markov) equilibrium by looking at an interpretation of the environment as a game between the saver and its future selves. We implement the equilibrium conditions in loans markets as auxiliary restrictions faced by the borrower.

- Our characterization yields a pair of functional equations that use auxiliary functions: when to default, and how much to borrow.

1. The determination of the defaulting threshold as an indifference between defaulting and not defaulting.

2. A Generalized Euler Equation (GEE) that determines the saving decision and where the various effects are weighted. This equation includes derivatives of the decision rules as in Krusell, Kuruşçu, and Smith (2002), and Klein, Krusell, and Ríos-Rull (2008). We look for differentiable decisions.

What Economies we look at

- To illustrate the approach we look at a variety of model economies and show how the method works in each of them and what they deliver. The economies that we look at are

1. The canonical default problem with short term debt only.
2. The canonical default problem with long term debt only.
3. A multiple maturity debt problem, (Arellano and Ramanarayanan (2012)).
4. A model of partial default (Arellano, Mateos-Planas, and Ríos-Rull (2013)).

What are the effects that we isolate? I Short term debt

- For short term debt, the GEE weighs the traditional expected marginal utility tomorrow (in those states of the world where there is no default) against two effects today:

  1. The traditional marginal utility of consumption today that is associated to a change in the savings *multiplied* by the probability of defaulting tomorrow (internalized by the market).

  2. Additional borrowing increases the set of states over which there is default tomorrow and this deteriorates the terms of the loans. This term involves the derivative of the defaulting decision with respect to debt size. In the presence of commitment this term would be absent.

II Long term debt adds two terms:

1. Borrowing more today reduces the continuation value of the debt due to a higher probability of default. It is the value of the debt at the defaulting threshold (bounded away from zero), times the density times the derivative of the default function at the amount borrowed.

2. Borrowing more today induces additional borrowing tomorrow that dilutes the continuation value of the debt. This term is the expected value of the derivative of the price function times the derivative of the savings function. This last effect is actively discussed in the literature. (Arellano and Ramanarayanan (2012), and others. Gomes, Jermann, and Schmid (2014)) pose the FOC which yields price derivatives. They do not get rid of these price derivatives. Instead, they differentiate again which yields second price derivatives (as the perturbation method in Klein, Krusell, and Ríos-Rull (2008)).

- We provide a formula for the derivatives of the price with respect to borrowing that we interpret as the expected value of the time inconsistency normalized by the marginal utility today. We think that there is no such analysis in the literature.

Comparison with Commitment

- We also provide a recursive characterization of the problem under commitment.

- Even with commitment, debt occurs in equilibrium, but in different circumstances than in the absence of commitment (probably with lower probability for a given amount obtained borrowing). (To compare with Adam and Grill (2012)).

- This yields a clear comparison of the issues that arise and can provide a base to assess what the is value of commitment.

III Coexistence of Short and Long Term Debt

Arellano and Ramanarayanan (2012)

- The relative attractiveness of both types of debt depends on subtle interactions between the values of the three states. Current long term debt, current short term debt, and the endowment.
- We are not yet ready to say much about how it works, even if we have characterized the relevant functional equations.

IV Partial default with incomplete debt discharge

Arellano, Mateos-Planas, and Ríos-Rull (2013)

- This environment provides a clear difference with the standard default problem:

  1. There is no proper default as debts do not disappear. Yet the borrower chooses the amount to not pay unilaterally.

  2. The unpaid amount carries over at a different rate (lower) than the standard debt, and right after not paying a certain fraction of output is lost.

- As a consequence this environment provides two forms of borrowing: a standard or voluntary, and an involuntary one with a fixed, low rate of return and an output loss penalty.

- In addition to assessing the trade-offs and the decision making, the GEE provides a comparison between the rewards to saving in each of the two forms.

Other Extensions

- We show how to pose Markov processes for the shock. It is trivial.

- Less dramatic punishment: Temporary exclusion of borrowing and possibility to save.

The canonical default model: Main features

- Long term uncontingent debt b with decay rate λ. This period it has to pay b, and next period it has an obligation to pay $(1-\lambda)b$ plus whatever additional debt it issues at equilibrium price Q. If $b < 0$ its rate is the risk free rate.

- Irreversible default: Once the agent defaults it reverts to autarky.
  We make this assumption to avoid cumbersome, uninteresting, record keeping notation. The extension to forgiveness after some suitable waiting time, and to being able to save while in autarky is immediate, yet garrulous.

Preferences, endowments, markets, commitment

- The agent, sovereign, government, has standard utility function $u(c)$ and discount rate $\beta < R^{-1}$.
- Endowment each period ϵ is iid with density $f(\epsilon)$ and c.d.f. $F(\epsilon)$.
- There are only uncontingent bonds, with many risk neutral borrowers at the risk free gross interest rate $R = 1 + r$.
- The agent cannot commit to anything. In particular it cannot commit to the circumstances under it will choose to default in the future, which could have been a form of contingency.

Posing the Environment Recursively

- We pose the environment recursively to focus on differential policy functions. This allows us to look at Markov equilibria that are the limit of finite economies. (Krusell, Kuruşçu, and Smith (2002)).
- The agent takes as given the decision *rules* of its future self.
- Long term debt is b. This is the amount to be paid per period and it decays at rate λ. Its price is $Q(b')$. The decision rule that determines how much to borrow is decision rule is denoted h, $b' = h(\epsilon, b)$.
- The default location is ϵ^* with decision rule denoted $\epsilon^* = d(b)$.

The agent's problem given future behavior d and h

$$v(\epsilon, b) = \max \left\{ u(\epsilon) + \beta \overbrace{\frac{1}{1-\beta} \int u(\epsilon)\, f(d\epsilon')}^{\overline{v}}, \qquad \text{default} \right.$$

$$\left. \max_{b'} \; u(c) + \beta \int^{d(b')} \left[u(\epsilon') + \beta \overline{v} \right] \, f(d\epsilon') + \beta \int_{d(b')} v(\epsilon', b')\, f(d\epsilon') \; \text{not} \right\}$$

$$\text{s.t.} \qquad c \leq \epsilon - b + Q(b') \left[b' - (1-\lambda) b \right]$$

- $Q(b')$ is the price of debt today when b' is chosen.

Zero profit condition of the price of debt

- In equilibrium, one unit of debt is worth the expected discounted value of its repayment plus its continuation value:

$$Q(b') = R^{-1} \left\{ [1 - F(d(b'))] + (1-\lambda) \int_{d(b')} Q[\overbrace{h(\epsilon', b')}^{b''}] \; f(d\epsilon') \right\}$$

Default Threshold

The household defaults when it is worth to do it

$$v[d(b), b] = u[d(b)] + \beta\ \overline{v}.$$

The FOC and envelope conditions

$$u_c(c)\ \{Q(b') + Q_b(b')[b' - (1-\lambda) b]\} = \beta \int_{d(b')} v_b(\epsilon', b')\ f(d\epsilon')$$

$$v_b(\epsilon, b) = u_c(c)\ \{1 + (1-\lambda)\ Q[h(\epsilon, b)] +$$

$$Q_b[h(\epsilon, b)]\ h_b(\epsilon, b)\ [(1-\lambda) b - h(\epsilon, b)] - Q[h(\epsilon, b)] h_b(\epsilon, b)\}$$

$$+ \beta\ h_b(\epsilon, b)\ \int_{d(h(\epsilon, b'))} v_b[\epsilon', h(\epsilon, b)]\ \ f(d\epsilon').$$

- Lines 2 and 3 of the envelope condition cancel by the FOC:

$$h_b(\epsilon, b)\Bigg[u_c(c)\ \{Q_b[h(\epsilon, b)]\ [(1-\lambda) b - h(\epsilon, b)] - Q[h(\epsilon, b)]\}$$

$$+ \beta\ \int_{d(h(\epsilon, b'))} v_b[\epsilon', h(\epsilon, b)]\ \ f(d\epsilon')\Bigg] = 0$$

This yields in compact notation

$$u_c\left\{Q - Q_{b'}\left[(1-\lambda)b - h\right]\right\} = \beta \int_{d'} \left\{ u'_c \left\{1 + (1-\lambda)\, Q'\right\} \right\} f(d\epsilon')$$

- But this object has $Q_{b'}$, the derivative of the pricing function evaluated at the amount of savings chosen. This is the object that we want to avoid.

Let us recap what we have so far

1. We have the FOC, in compact notation

$$u_c \left\{Q - Q_{b'}[(1-\lambda)b - h]\right\} = \beta \int_{d'} u'_c \left[1 + (1-\lambda)\, Q'\right]\, f(d\epsilon'),$$

2. In less-compact notation, the equation that determines d

$$v[d(b), b] = u[d(b)] + \frac{\beta}{1-\beta} \int_0^\infty u(\epsilon')\ f(d\epsilon'),$$

3. The definition of prices Q (note that its derivatives will involve terms with future derivatives as well, a problem).

$$Q(b') = R^{-1} \left\{ [1 - F(d(b'))] + (1-\lambda) \int_{d(b')} Q[h(\epsilon', b')]\ f(d\epsilon') \right\}.$$

Short term debt $\lambda = 1$, is easy to solve

- Given that debt prices do not include its future values:

$$Q \equiv Q(b') = R^{-1}\,[1 - F(d')],$$

- Neither does its derivative

$$Q_{b'} \equiv Q_b(b') = -R^{-1}\,f(d')\,d'_{b'} = -R^{-1}\,F_d(d')\,d'_{b'},$$

- Which allows us to rewrite the FOC as a GEE

$$u_c\,[(1 - F(d')) - f(d')\,d'_{b'}\,h] = \beta\,R\,\int_{d'} u'_c\,f(d\epsilon').$$

Summarizing: short term debt equilibrium funct equations

- The equation that determines the default threshold (indifference between defaulting and not defaulting)

$$v[d(b), b] = u[d(b)] + \beta\ \overline{v},$$

- The GEE

$$u_c \left\{ \begin{array}{ll} [(1 - F(d'))] & \text{per unit gain in today's consumption} \\ -f(d')\ d'_{b'}\ h & \text{reduction of the price of debt} \end{array} \right\} = \beta\ R \int_{d'} u'_c\ f(d\epsilon').$$

- An auxiliary object: the definition of value function

$$v(\epsilon, b) = u[\epsilon - b + h(\epsilon, b)(1 - F(d(h(\epsilon, b))))] + \beta \int v(\epsilon', h(\epsilon, b)) f(d\epsilon)$$

Rewriting the GEE with arguments

$$u_c(\epsilon + b - Q[h(\epsilon, b)])\Big\{[1 - F(d[h(\epsilon, b)])] -$$

$$- f(d[h(\epsilon, b)]) \;\; d_b[h(\epsilon, b)] \;\; h(\epsilon, b)\Big\} =$$

$$\beta \, R \int_{d[h(\epsilon,b)]} u_c'(\epsilon' + h(\epsilon, b) - Q[\overbrace{h(\epsilon', h(\epsilon, b))}^{b''}]) \, f(d\epsilon').$$

Compare with the Commitment Case. It is also recursive

- Write it as a commitment to repay k in expected value. The agent can choose to default. It chooses how much to pay m, and with what probability, $[1 - F(\epsilon^c)]$.

- m with commitment compares to b without.

$$v^c(k) = \max_{\substack{m,\epsilon^c,\\ c(\epsilon),k'(\epsilon)}} \left\{ \int_0^{\epsilon^c} \widehat{v}(\epsilon) f(d\epsilon) + \int_{\epsilon^c} u[c(\epsilon)] f(d\epsilon) + \beta \int_{\epsilon^c} v^c[k'(\epsilon)] f(d\epsilon) \right\}$$

subject to

$$\widehat{v}(\epsilon) = u(\epsilon) + \beta \overline{v}, \qquad \text{punishment to autarky}$$

$$k = [1 - F(\epsilon^c)] m, \qquad \text{repayment}$$

$$c(\epsilon) = \epsilon + \frac{k'(\epsilon)}{1+r} - m, \quad \epsilon > \epsilon^c. \qquad \text{budget constraint}$$

Rewriting it compactly, getting the FOCs

$$v^c(k) = \max_{\epsilon^c, k'(\epsilon)} \left\{ \int_0^{\epsilon^c} \left(u(\epsilon) + \beta \overline{v} \right) f(d\epsilon) + \right.$$
$$\left. \int_{\epsilon^c} u \left[\epsilon + \frac{k'(\epsilon)}{1+r} - \frac{k}{1-F(\epsilon^c)} \right] f(d\epsilon) + \beta \int_{\epsilon^c} v^c[k'(\epsilon)] \, f(d\epsilon) \right\}.$$

FOC wrt to $k'(\epsilon)$:

$$u_c[c(\epsilon)] + \beta(1+r) v_k^c[k'(\epsilon)] = 0.$$

FOC wrt to ϵ^c:

$$u(\epsilon^c) + \beta \overline{v} = u[c(\epsilon^c)] + \beta v^c[k'(\epsilon^c)] + \int_{\epsilon^c} u_c[c(\epsilon)] \frac{k}{[(1-F(\epsilon^c)]^2} \, f(d\epsilon).$$

Using the envelope condition under commitment

$$v_k^c(k) = -\frac{\int_{\epsilon^c} u_c[c(\epsilon)]\; F(d\epsilon)}{1 - F(\epsilon^c)}$$

Putting it forward and using the decision rules

$$v_k^c[h^c(\epsilon, k)] = -\frac{\int_{d^c[h(\epsilon,k)]}\; u_c[c^c(h^c(\epsilon, k), \epsilon')]\; F(d\epsilon')}{1 - F(d^c[h^c(\epsilon, k)])}$$

Combining the FOC wrt to $k'(\epsilon)$ and the envelope condition

$$u_c[c^c(\epsilon, k)]\Big(1 - F(d^c[h^c(\epsilon, k)])\Big) =$$
$$\beta(1+r)\int_{d^c[h^c(\epsilon,k)]}\; u_c[c^c(d^c[h^c(\epsilon, k), \epsilon')]\; F(d\epsilon').$$

The functional equations that characterize the problem

$$u[d^c(k)] + \beta\overline{v} = u\left[d^c(k) + \frac{h^c(\epsilon^c, k)}{1+r} - \frac{k}{1-F[d^c(k)]}\right] + \beta v[h^c(\epsilon^c, k)] + \int_{\epsilon^c} u_c\left[\epsilon + \frac{h^c(\epsilon, k)}{1+r} - \frac{k}{1-F[d^c(k)]}\right] f(d\epsilon)$$

$$u_c\left[\epsilon + \frac{h^c(\epsilon, k)}{1+r} - \frac{k}{1-F[d^c(k)]}\right]\Big(1 - F(d^c[h^c(\epsilon, k)])\Big) = \beta(1+r)\int_{d^c[h^c(\epsilon,k)]} u_c\left[\epsilon' + \frac{h^c[\epsilon', h^c(\epsilon, k)]}{1+r} - \frac{h^c(\epsilon, k)}{1-F(d^c[h^c(\epsilon, k)])}\right] f(d\epsilon').$$

Compactly,

$$\begin{aligned} u[\epsilon^c] + \beta\overline{v} &= u\left[c^c(\epsilon^c, k)\right] + \beta v[h^c(\epsilon^c, k)] + \frac{k}{(1-F[d^c(k)])^2}\int_{\epsilon^c} u_c' \, f(d\epsilon), \\ u_c\left[1 - F(d'^c)\right] &= \beta(1+r)\int_{d'^c} u_c \, F(d\epsilon'). \end{aligned}$$

Comparison between commitment and no commitment

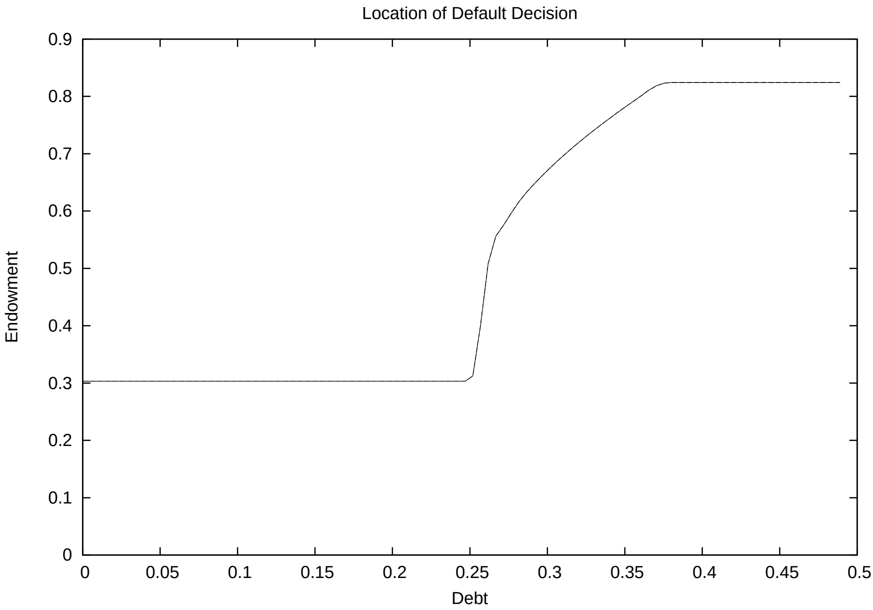
The value equation

$$\begin{array}{ll} \text{W} & u[\epsilon^c] + \beta \overline{v} = u\left[c^c(\epsilon^c, k)\right] + \beta v^c[h^c(\epsilon^c, k)] + \dfrac{k \int_{\epsilon^c} u_c \, f(d\epsilon)}{(1 - F[d^c(k)])^2} \\ \text{Wo} & u[\epsilon^*] + \beta \overline{v} = u\left[c(\epsilon^*, b)\right] + \beta v[h(\epsilon^*, b)] \end{array}$$

The GEE

$$\begin{array}{llll} \text{With} & u_c \left[1 - F(d'^c)\right] & = & \beta \, (1 + r) \displaystyle\int_{\epsilon^c} u'_c \, f(d\epsilon') \\ \text{Without} & u_c[1 - F(d')] - u_c \, f(d') \, d'_{b'} \, h & = & \beta \, (1 + r) \displaystyle\int_{\epsilon^*} u'_c \, f(d\epsilon'). \end{array}$$

- The arguments b and k are not strightly comparable, but $b[1 - F(d(b))]$ and k are comparable.


Location of Default Decision
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
Endowment
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
Debt

II Long Term Bonds, $\lambda < 1$. Harder

Recall the FOC of this problem

$$u_c\{Q + Q_{b'}[h - (1-\lambda)b]\} = \beta \int_{d'} \{u_c'[1 + (1-\lambda)\, Q']\} f(d\epsilon'),$$

with its associated price Q and its derivative $Q_{b'}$

$$\begin{aligned} Q &= R^{-1}\left\{(1 - F(d')) + (1-\lambda)\int_{d'} Q' \;\; f(d\epsilon')\right\}, \\ Q_{b'} &= R^{-1}\left\{-F_d(d')d'_{b'} + (1-\lambda)\left[-d'_{b'}\widetilde{Q}' + \int_{d'} Q'_{b''}\; h'_{b'}\; f(d\epsilon')\right]\right\}, \end{aligned}$$

where we denote with $\widetilde{Q}'$ the price at the default threshold as

$$\widetilde{Q}' \equiv Q[h(d(b'), b')]$$

The task is to eliminate $Q_{b'}$: use FOC and move forward

$$Q_{b'} = \mathcal{B}(h, d', Q, Q') \equiv \frac{\beta \int_{d'} \{u_c'[1 + (1-\lambda)\, Q']\} f(d\epsilon') + Q u_c}{u_c[(1-\lambda)b - h]}.$$

- Put this forward (for $Q_{b''}$) a in the explicit derivation of $Q_{b'}$ using the equilibrium condition for prices (third equation in previous page) and

$$Q_{b'} = R^{-1}\left\{-F_d(d')d_{b'}' + (1-\lambda)\left[-d_{b'}'\widetilde{Q}' + \int_{d'} \mathcal{B}(h', d'', Q', Q'')h_{b'}' f(d\epsilon')\right]\right\}$$

- Substituting it back into the FOC yields

$$0 = u_c\Bigg\{Q + [h - (1-\lambda)b]R^{-1}\Bigg\{-F_d(d')\; d_{b'}' + (1-\lambda) \left[-d_{b'}'\widetilde{Q}' + \int_{d'} \mathcal{B}(h', d'', Q', Q'')\; h_{b'}'\; f(d\epsilon')\right]\Bigg\}\Bigg\} - \beta \int_{d'} \{u_c'[1 + (1-\lambda)\, Q']\}\; f(d\epsilon').$$

So we get 2 functl eqns h, d that use auxiliary fns $Q, v, \mathcal{B}$

- Auxiliary functions (Q and v look like contractions).

$$Q(h(\epsilon, b); h, d) = \quad R^{-1}\Big\{(1 - F[d'(h(\epsilon, b))])+$$

$$(1-\lambda)\int_{d'(h(\epsilon,b))} Q[h(\epsilon', h(\epsilon, b)); h, d] f(d\epsilon')\Big\}$$

$$v(\epsilon, b; h, d) = \quad \max\Big\{u(\epsilon) + \frac{\beta}{1-\beta}\widetilde{v},$$

$$u(\epsilon + b - h(\epsilon, b)) + \beta \int v[\epsilon', h(\epsilon, b'; h, d)]\ f(d\epsilon')\Big\}$$

$$\mathcal{B}(\epsilon, b; h, d) \equiv \quad \frac{\beta \int_{d'} \{u'_c[1 + (1-\lambda)\ Q']\} f(d\epsilon') + Qu_c}{u_c[(1-\lambda)b - h]}$$

Equilibrium functional equations

$$0 = u_c \Big\{ Q + [h - (1-\lambda) b] R^{-1} \Big\{ -F_d(d')\, d'_{b'} + (1-\lambda) \Big[-d'_{b'} \widetilde{Q}' + \int_{d'} \mathcal{B}(h', d'', Q', Q'')\, h'_{b'}\, f(d\epsilon') \Big] \Big\} \Big\} - \beta \int_{d'} \{ u'_c [1 + (1-\lambda)\, Q'] \}\, f(d\epsilon'),$$

$$v[d(b), b] = u[d(b)] + \beta \int v(h(\epsilon, b') f(d\epsilon').$$

Rewriting all objects explicitly using functions d and h

- Auxiliary functions

$$Q(b';h,d) = R^{-1}\left\{(1-F(d(b'))) + (1-\lambda)\int_{d(b')} Q(h(\epsilon',b');h,d)f(d\epsilon')\right\}$$

$$u_c(\epsilon,b;h,d) \equiv \frac{\mathrm{d}\ u(\epsilon+b+Q(h(\epsilon,b);h,d)((1-\lambda)b-h(\epsilon,b)))}{\mathrm{d}\ c}$$

$$\begin{aligned}\mathcal{B}(\epsilon,b;h,d) \equiv \{\beta\int_{d(h(\epsilon,b))} u_c(\epsilon',h(\epsilon,b);h,d)[1+(1-\lambda)\,Q(h(\epsilon',h(\epsilon,b));h,d)]f(d\epsilon') \\ + Q(h(\epsilon,b);h,d)u_c(\epsilon,b;h,d)\} \\ /\{[(1-\lambda)b-h(\epsilon,b)]u_c(\epsilon,b;h,d)\}\end{aligned}$$

$$\begin{aligned}v(\epsilon,b;h,d) = \max\Big\{u(\epsilon)+\frac{\beta}{1-\beta}\int u(\epsilon')f(d\epsilon'), \\ u(\epsilon+b+Q(h(\epsilon,b);h,d)((1-\lambda)b-h(\epsilon,b)))+\beta\int v(\epsilon',h(\epsilon,b);h,d)f(d\epsilon')\Big\}\end{aligned}$$

- Equilibrium functional equations

$$
\begin{aligned}
0 \quad = \quad & u_c(\epsilon, b; h, d)\Bigg\{ Q(h(\epsilon, b); h, d) + [h(\epsilon, b) - (1-\lambda)b]R^{-1} \\
& \Bigg\{ -F_d(d(h(\epsilon, b)))d_b(h(\epsilon, b)) \\
& +(1-\lambda)\Bigg[-d_b(h(\epsilon, b))Q(h(h(\epsilon, b), d(h(\epsilon, b))); h, d) \\
& + \int_{d(h(\epsilon,b))} \mathcal{B}(\epsilon', h(\epsilon, b); h, d) h_b(\epsilon', h(\epsilon, b)) f(d\epsilon') \Bigg] \Bigg\} \Bigg\} \\
& -\beta \int_{d(h(\epsilon,b))} u_c(\epsilon', h(\epsilon, b); h, d)[1 + (1-\lambda)\, Q(h(\epsilon', h(\epsilon, b)); h, d)] f(d\epsilon')
\end{aligned}
$$

$$
v[d(b), b] \quad = \quad u[d(b)] + \beta \int v(h(\epsilon, b')\; f(d\epsilon')
$$

Isolating effects using compact notation

$Q = Q(h(\epsilon, b); h, d)$, $Q' = Q[h(\epsilon', h(\epsilon, b)); h, d]$, $\widetilde{Q}' = Q[h(h(\epsilon, b), d(h(\epsilon, b))); d, h]$, $d = d(b)$, $d' = d(h(\epsilon, b))$, $\mathcal{B}' = \mathcal{B}(\epsilon', h(\epsilon, b); h, d)$, $h'_b = h_b(\epsilon', h(\epsilon, b))$. Then

$$
\begin{aligned}
u_c \Bigg\{ Q\ R + \quad & \text{consumption gain} \\
[h - (1-\lambda) b] \quad & \text{new borrowing times} \\
\Bigg(-f(d')\ d'_b \quad & \text{tomorrow's payment loss} \\
+ (1-\lambda) \Bigg[-d'_b\ \ \widetilde{Q}'\ f(d') \quad & \text{tomorrow's principal loss} \\
+ \int_{d'} \mathcal{B}' h'_b f(d\epsilon') \Bigg] \Bigg) \Bigg\} \quad & \text{dilution due to additional debt} \\
= \beta\ R \int_{d'} u'_c\ [1 + (1-\lambda)\ Q']\ f(d\epsilon'). &
\end{aligned}
$$

A closer peek at the effects of long term debt

1. Additional borrowing induces a capital loss in amount $\widetilde{Q}[d(h)]$,

$$(1-\lambda)\left[-d_b' \;\; \widetilde{Q}[d(h)] \; f(d')\right]$$

2. The dilution term is more contrived,

$$(1-\lambda)\int_{d'} \frac{\beta \int_{d''}\{u_c''[1+(1-\lambda)\,Q'']\} f(d\epsilon'') + Q'\,u_c'}{u_c'[(1-\lambda)h - h']} \; h_b' \; f(d\epsilon')$$

- It is the surviving fraction of the debt times the expect value of the harm that additional debt does. Such damage is the term in the ratio. We can think of it as the expected amount of the time inconsistent term of the FOC tomorrow (the difference between the FOC with and without commitment) normalized by the marginal utility times the amount borrowed tomorrow.

With Commitment Long and Short Term Debt is the Same

$$v^c(k) = \max_{\substack{m,\epsilon^c,\\ c(\epsilon),k'(\epsilon)}} \left\{ \int_0^{\epsilon^c} (u(\epsilon) + \beta \overline{v}) f(d\epsilon) + \int_{\epsilon^c} (u[c(\epsilon)] + \beta v^c[k'(\epsilon)]) f(d\epsilon) \right\}$$

$$\text{s.t} \qquad k + \frac{1-\lambda}{r+\lambda} k = [1 - F(\epsilon^c)] m$$

$$c(\epsilon) = \epsilon + \frac{k'(\epsilon)}{r+\lambda} - p, \qquad \text{when } \epsilon > \epsilon^c$$

$$v^c(k) = \max_{\epsilon^c, k'(\epsilon)} \left\{ \int_0^{\epsilon^c} \left(u(\epsilon) + \beta \overline{v} \right) f(d\epsilon) + \int_{\epsilon^c} u \left[\epsilon + \frac{k'(\epsilon)}{r+\lambda} - \frac{k \frac{1+r}{r+\lambda}}{1 - F(\epsilon^c)} \right] f(d\epsilon) + \beta \int_{\epsilon^c} v^c[k'(\epsilon)] f(d\epsilon) \right\}$$

The first order condition with respect to $k'(\epsilon)$ and ϵ^c are

$$\begin{aligned} u_c(\epsilon) &= -\beta(r+\lambda) v_k^c[k'(\epsilon)] \\ u(\epsilon^c) + \beta \overline{v} &= u[c(\epsilon^c)] + \beta v[k'(\epsilon^c)] + \int_{\epsilon^c} u_c[c(\epsilon)] \frac{k \frac{1+r}{r+\lambda}}{[(1-F(\epsilon^c)]^2} f(d\epsilon) \end{aligned}$$

The envelop condition with respect to k gives

$$v_k^c(k) = -\frac{1+r}{r+\lambda} \frac{\int_{\epsilon^c} u_c[c(\epsilon)] f(d\epsilon)}{1-F(\epsilon^c)}$$

Let $\epsilon^c = d^c(k)$, then forwarding the envelop condition yields

$$v_k^c[k'(\epsilon)] = -\frac{1+r}{r+\lambda} \frac{\int_{d[k'(\epsilon)]} \; u_c[c(\epsilon')] \; f(d\epsilon')}{1-F(d^c[k'(\epsilon)])}$$

Long Term Debt With Commitment

Combining the FOC wrt $k'(\epsilon)$ and the envelop condition yields

$$u_c[c(\epsilon)]\left(1 - F(d^c[k'(\epsilon)])\right) = \beta(1+r)\int_{d^c[k'(\epsilon)]} u_c[c(\epsilon')]\, f(d\epsilon')$$

Let $k' = h^c(\epsilon, k)$ and $c^c(\epsilon, k) = \epsilon + \frac{h^c(\epsilon,k)}{1+r} - \frac{k\frac{1+r}{r+\lambda}}{1-F[\epsilon^c]}$ then

$$\begin{aligned} u(\epsilon^c) + \beta\overline{v} = u\left[c^c(\epsilon^c, k)\right] \quad & +\beta v^c(h^c) \\ & + \int_{\epsilon^c} u_c\left[c^c(\epsilon^c, k)\right] \frac{k\frac{1+r}{r+\lambda}}{(1-F[\epsilon^c])^2}\, f(d\epsilon), \\ u_c\left[c^c(\epsilon, k)\right]\left[1 - F(d'^c)\right] \quad & = \beta(1+r)\int_{d'^c} u_c\left[c^c(\epsilon', h)\right]\, f(d\epsilon'). \end{aligned}$$

Which coincides with short term commitment when $\lambda = 1$. QED

Comparison bw commitment and no commitment

$$\text{W} \quad u(\epsilon^c) + \beta\overline{v} = \quad u\left[c^c(\epsilon^c, b)\right] + \beta v(h^c) + \frac{b\frac{1+r}{r+\lambda}}{(1-F[\epsilon^c])^2} \int_{\epsilon^c} u'_c \; f(d\epsilon),$$

$$\text{Wo} \quad u[\epsilon^*] + \beta \; \overline{v} = \qquad u\left[c(\epsilon^*, a)\right] + \beta v[h(\epsilon^*, b)]$$

The GEE

$$\text{W} \qquad u_c \left[1 - F(d'^c)\right] \;\; = \;\; \beta \left(1 + r\right) \int_{\epsilon^c} u'_c \; f(d\epsilon')$$

$$\text{Wo} \quad u_c[h - (1-\lambda)b]Q \; R \;\; = \;\; \beta\left(1+r\right) \int_{d'} u'_c \left[1 + (1-\lambda)\, Q'\right] f(d\epsilon')$$

$$+ u_c[h - (1-\lambda)b]\left\{ f(d')\; d'_b - (1-\lambda)\left[-d'_b \;\; \widetilde{Q}' \; f(d') + \int_{d'} \mathcal{B}' h'_b f(d\epsilon')\right]\right\}.$$

Coexistence of Short and Long Term Debt

Coexistance of Short and Long Term Debt

- Irreversible default, with punishment being autarky with value $\widehat{v}(\epsilon)$ and expected value $\overline{v}$. iid endowment ϵ with density f and cdf F.
- Long term debt is a console ($\lambda = 0$).
- a and P one-period debt and its price; b and Q long-term debt. Decision rules: $d(a, b)$ default threshold; $a' = g(\epsilon, a, b)$ and $b' = h(\epsilon, a, b)$.
- The budget constraint is

$$c = \epsilon + P(a', b')a' + Q(a', b')(b' - b) - a - b$$

- If there was no default, one unit of a' yields today R^{-1} units of the good today while one unit of b' yields $(R-1)^{-1}$.

The model

$$v(\epsilon, a, b) = \max_{a', b'} \;\; u(c) + \beta \int_0^{d(a',b')} \widehat{v}(\epsilon')\, f(d\epsilon') + \beta \int_{d(a',b')} v(\epsilon', a', b')\, f(d\epsilon')$$

s.t.

$$c = \epsilon + P(a', b')a' + Q(a', b')(b' - b) - a - b$$

- Default threshold $d(a, b)$ is $v(d(a, b), a, b) = \widehat{v}(d(a, b))$. Prices

$$\begin{aligned} P(a', b') &= R^{-1}[1 - F(d(a', b'))] \\ Q(a', b') &= R^{-1}\Big\{[1 - F(d(a', b'))] \\ &\qquad + \int_{d(a',b')} Q(g(\epsilon', a', b'), h(\epsilon', a', b'))\, f(d\epsilon')\Big\} \end{aligned}$$

FOC and Envelope

$$\begin{aligned}
0 &= u_c[P + a'P_a + (b'-b)Q_a] - \beta \int_{d(a',b')} v_a(\epsilon', a', b') f(\mathrm{d}\epsilon') \\
0 &= u_c[Q + (b'-b)Q_b + a'P_b] - \beta \int_{d(a',b')} v_b(\epsilon', a', b') f(\mathrm{d}\epsilon') \\
v_a(\epsilon, a, b) &= -u_c \\
v_b(\epsilon, a, b) &= -u_c(1 + Q(a', b'))
\end{aligned}$$

- Substitute back (using $g = a' = g(\epsilon, a, b)$ and $h = b' = h(\epsilon, a, b)$)

$$\begin{aligned}
u_c[P + gP_a + (h-b)Q_a] - \beta \int_{d(g,h)} u_c' f(\mathrm{d}\epsilon') &= 0 \\
u_c[Q + (h-b)Q_b + gP_b] - \beta \int_{d(g,h)} (1 + Q') u_c' f(\mathrm{d}\epsilon') &= 0
\end{aligned}$$

Use P and Q to find derivatives in FOC

Recall that prices are given by

$$P(a', b') = R^{-1}[1 - F(d(a', b'))]$$
$$Q(a', b') = R^{-1}\left\{[1 - F(d(a', b'))] + \int_{d(a',b')} Q(g(\epsilon', a', b'), h(\epsilon', a', b'))\, f(d\epsilon')\right\}$$

Directly differentiating P and Q above wrt a and b:

$$\begin{aligned}
P_a &= R^{-1}(-F_d d'_a) \\
P_b &= R^{-1}(-F_d d'_b) \\
Q_a &= R^{-1}\left\{-F_d(d') d'_{a'} + \left[-d'_{a'}\widetilde{q}' + \int_{d'} \left[Q'_a g'_a + Q'_b h'_a\right] f(d\epsilon')\right]\right\}, \\
Q_b &= R^{-1}\left\{-F_d(d') d'_{b'} + \left[-d'_{b'}\widetilde{q}' + \int_{d'} \left[Q'_a g'_b + Q'_b h'_b\right] f(d\epsilon')\right]\right\},
\end{aligned}$$

Use tomorrow's FOC to pin down Q'_a and Q'_b

- Define

$$
\begin{aligned}
(h-b)\ \mathcal{A}(\epsilon, a, b) &\equiv \left[\frac{\beta \int_{d(g,h)} u'_c f(d\epsilon') - P u_c}{u_c} - g\ P_a\right] \\
(h-b)\ \mathcal{B}(\epsilon, a, b) &\equiv \left[\frac{\beta \int_{d(g,h)} u'_c (1+Q') f(d\epsilon') - Q u_c}{u_c} - g\ P_b\right]
\end{aligned}
$$

- From the FOC it turns out that

$$
\begin{aligned}
Q'_a &= \mathcal{A}(\epsilon', g, h) \\
Q'_b &= \mathcal{B}(\epsilon', g, h)
\end{aligned}
$$

In sum: the GEE

- The FOC's

$$\begin{aligned} u_c[P + gP_a + (h-b)Q_a] &= \beta \int_{d(g,h)} u_c' f(\mathrm{d}\epsilon') \\ u_c[Q + gP_b + (h-b)Q_b] &= \beta \int_{d(g,h)} (1+Q') u_c' f(\mathrm{d}\epsilon') \end{aligned}$$

- ... with price derivatives given as

$$P_a = R^{-1}(-F_d d_a')$$

$$P_b = R^{-1}(-F_d d_b')$$

$$Q_a = R^{-1}\left\{-F_d(d')d_{a'}' + \left[-d_{a'}'\widetilde{q}' + \int_{d'} \left[Q_a' g_a' + Q_b' h_a'\right] f(d\epsilon')\right]\right\},$$

$$Q_b = R^{-1}\left\{-F_d(d')d_{b'}' + \left[-d_{b'}'\widetilde{q}' + \int_{d'} \left[Q_a' g_b' + Q_b' h_b'\right] f(d\epsilon')\right]\right\},$$

Analysis of the GEEs

- Note that the FOC's tell us that the optimal choice of each type of debt takes into account, not only what is directly obtained when issuing, but also the induced changes in the prices of both types of debt. To understand what is involved requires more detailed expressions

- With respect to the effects on the price of short term debt,

$$P_a = R^{-1}(-f\ d'_a)$$

$$P_b = R^{-1}(-f\ d'_b)$$

- We see that the difference between the two relates only to the effect that each type of debt has on the probability of default as indicated by how much each type of debt moves the default threshold.

Effects on long term debt prices of both types of debt

$$Q_a = R^{-1}(-F_d d'_a) + R^{-1}E[\mathcal{A}(\epsilon', g, h)g'_a + \mathcal{B}(\epsilon', g, h)h'_a]$$

$$Q_b = R^{-1}(-F_d d'_b) + R^{-1}E[\mathcal{A}(\epsilon', g, h)g'_b + \mathcal{B}(\epsilon', g, h)h'_b]$$

$$\mathcal{A}(\epsilon', a', b') \equiv \left\{ \left[\frac{\beta \int_{d(g',h')} u''_c f(d\epsilon'') - P' u'_c}{u'_c} - g' R^{-1}(-f(d'')\ d''_a) \right] \frac{1}{h' - h} \right\}$$

$$\mathcal{B}(\epsilon', a', b') \equiv \left\{ \left[\frac{\beta \int_{d(g',h')} u''_c (1 + Q'') f(\mathrm{d}\epsilon'') - Q' u'_c}{u'_c} - g' R^{-1}(-f(d'')\ d'_b) \right] \frac{1}{h' - h} \right\}$$

- As you can imagine, we still have to digest these terms to relate them to price sensitivity in certain contexts (Arellano and Ramanarayanan (2012)) .

The Model with Partial Default

(Arellano, Mateos-Planas, and Ríos-Rull (2013))

Partial default and its GEE

- What is not paid accumulates at rate $\overline{R}$, and reduces output tomorrow. Think of voluntary and involuntary borrowing from the point of view of the lenders.

- Endowment ϵ with density f and cdf F.

- Asset position is A, more precisely $A > 0$ is the amount to pay today.

- Unpaid debt is $0 \leq D \leq A$, it accumulates at exogenous rate $\overline{R}$ and it reduces the endowment tomorrow a fraction $[1 - \psi(D)]$.

- New emissions of (voluntary) debt are B, become part of A' one for one, and are priced at $Q(A, B, D)$.

Possing the model recursively, $b(\epsilon, a)$ and $d(\epsilon, a)$

$$
v(\epsilon, a) = \max_{b,d} \;\; u[\epsilon - (a-d) + Q(a,b,d)b] + \\
\beta E\{v(\epsilon'\psi(d), \lambda a + b + (1-\lambda)\overline{R}d)\}
$$

Remarks: $Q = \frac{R^{-1}}{(1-\lambda R^{-1})}$ when $b \leq 0$; we are ignoring in the text $d \leq a$;

- The FOC and envelope of this problem are

$$
\begin{aligned}
0 &= u_c[Q_b b + Q] + \beta \, E\{v'_a\} \\
0 &= u_c[1 + Q_d b] + \beta \, E\{(1-\lambda)\overline{R} v'_a + \epsilon' \psi_d v'_\epsilon\} \\
v_a &= u_c \left\{-1 + Q_a b\right\} + \beta\lambda \, E\left\{v'_a\right\} \;\; (\text{invoking optimality tomorrow}) \\
&= -u_c(1 + \lambda Q + b(\lambda Q_b - Q_a)) \;\; (\text{using 1st FOC}) \\
v_\epsilon &= u_c
\end{aligned}
$$

Possing the model recursively, $b(\epsilon, a)$ and $d(\epsilon, a)$

- Substitute back into the FOC's so they contain price derivatives Q_b, Q_d, Q'_b and Q'_a:

$$0 \quad = u_c[Q_b b + Q] - \beta E\{u'_c(1 + \lambda Q' + b'(\lambda Q'_b - Q'_a))\}$$

$$0 \quad = u_c[1 + Q_d b] + \beta E\{u'_c[\epsilon' \psi_d - (1-\lambda)\overline{R}(1 + \lambda Q' + b'(\lambda Q'_b - Q'_a))]\}$$

Strategy to derive the GEE

- We need to calculate price derivatives in FOC.

1. Define auxiliary price function Q via zero-profit condition.
2. Differentiate it to get price derivatives that depend on tomorrow's price derivatives
3. Solve tomorrow's price derivatives from the FOC shifted forward.
4. Build the GEE.

Step 1 - auxiliary function for prices

- The value of a claim to one unit of debt is

$$H(\epsilon, a) = \left(1 - \frac{d(\epsilon, a)}{a}\right) + \frac{1}{1+r}\left(\lambda + \overline{R}(1-\lambda)\frac{d(\epsilon, a)}{a}\right)$$
$$E\{H(\epsilon'\psi(d(\epsilon, a)), \lambda a + b(\epsilon, a) + (1-\lambda)\overline{R}d(\epsilon, a))\}.$$

- a zero profit condition determines the price function

$$Q(a, b, d) = \frac{1}{1+r}\ E\{H(\epsilon'\psi(d), \lambda a + b + (1-\lambda)\overline{R}d)\}.$$

- Combining

$$H(\epsilon, a) = \left(1 - \frac{d(\epsilon, a)}{a}\right) + \left(\lambda + \overline{R}(1-\lambda)\frac{d(\epsilon, a)}{a}\right) Q(a, b(\epsilon, a), d(\epsilon, a)).$$

- Substituting (ie, killing H) auxiliary function of prices is

$$\begin{aligned} Q(a, b, d) \quad = \quad & \frac{1}{1+r} E\left\{\left(1 - \frac{d(y', a')}{a'}\right)\right. \\ & \left. + \left(\lambda + \overline{R}(1-\lambda)\frac{d(\epsilon', a')}{a'}\right) Q(a', b(\epsilon', a'), d(\epsilon', a'))\right\}, \end{aligned}$$

$$\text{where} \quad \begin{aligned} \epsilon' \quad &= \quad \epsilon'\psi(d), \\ a' \quad &= \quad \lambda a + b + (1-\lambda)\overline{R}d. \end{aligned}$$

Step 2 - Use price function Q to find derivatives in FOC

- Q'_b and Q'_a drop from FOC since, obviously, $Q_a = \lambda Q_b$
- Wrt b and d:

$$
\begin{aligned}
Q_b(a,b,d) \quad = \quad & \frac{1}{1+r} E \left\{ -\frac{d_a(\epsilon',a')a' - d(\epsilon',a')}{a'^2} \right. \\
& + \left(\overline{R}(1-\lambda) \frac{d_a(\epsilon',a')a' - d(\epsilon',a')}{a'^2} \right) Q(a', b(\epsilon',a'), d(\epsilon',a')) \\
& \left. + \left(\lambda + \overline{R}(1-\lambda) \frac{d(\epsilon',a')}{a'} \right) [\lambda Q'_b + Q'_b b_a(\epsilon',a') + Q'_d d_a(\epsilon',a')] \right\} \text{ via } a'
\end{aligned}
$$

$$
\begin{aligned}
Q_d(a,b,d) \quad = \quad & \frac{1}{1+r} E \left\{ -\frac{d_\epsilon(\epsilon',a')\epsilon' \psi_d(d)}{a'} \right. \\
& + \left(\overline{R}(1-\lambda) \frac{d_\epsilon(\epsilon',a')\epsilon' \psi_d(d)}{a'} \right) Q(a', b(\epsilon',a'), d(\epsilon',a')) \\
& \left. + \left(\lambda + \overline{R}(1-\lambda) \frac{d(\epsilon',a')}{a'} \right) [Q'_b b_\epsilon(\epsilon',a') + Q'_d d_\epsilon(\epsilon',a')] \epsilon' \psi_d(d) \right\} \text{ via } \epsilon' \\
& + (1-\lambda)\overline{R} Q_b(a,b,d) \text{ via } a'\text{; see } Q_b(\ldots) \text{ above}
\end{aligned}
$$

... where short-hand notation stands for

$$
Q'_a \quad = \quad Q_1(a', b(\epsilon',a'), d(\epsilon',a')),\ Q'_b = Q_2(a', b(\epsilon',a'), d(\epsilon',a')),\ Q'_d = Q_3(a', b(\epsilon',a'), d(\epsilon',a')).
$$

Step 3 - Use tomorrow's FOC to pin down Q'_b and Q'_d

- define

$$\begin{aligned}
\mathcal{B}(\epsilon, a) &\equiv \frac{\beta E\{u'_c(1+\lambda Q')\} - Qu_c}{bu_c} \\
\mathcal{D}(\epsilon, a) &\equiv \frac{-\beta E\{u'_c(\epsilon' \psi_d - (1-\lambda)\overline{R}(1+\lambda Q'))\} - u_c}{bu_c}
\end{aligned}$$

... where short-hand notation stands for

$$\begin{aligned}
u_c &\equiv \frac{du}{dc}[\epsilon - (a - d(\epsilon, a)) + Qb(\epsilon, a)] \\
u'_c &\equiv \frac{du}{dc}[\epsilon' - (a' - d(\epsilon', a')) + Q' b(\epsilon', a')] \\
Q &= Q(b(\epsilon, a), d(\epsilon, a), a) \\
Q' &= Q(b(\epsilon', a'), d(\epsilon', a'), a') \\
\epsilon' &= \epsilon' \psi(d(\epsilon, a)) \\
a' &= \lambda a + b(\epsilon, a) + (1-\lambda)\overline{R} d(\epsilon, a)
\end{aligned}$$

- From the FOC (1) it turns out that

$$\begin{aligned}
Q'_b &= \mathcal{B}(\epsilon', a') \\
Q'_d &= \mathcal{D}(\epsilon', a')
\end{aligned}$$

Step 4 - Collecting pieces: the GEE

- The FOC

$$\begin{aligned} u_c[Q_b b + Q] - \beta E\{u'_c(1+\lambda Q')\} &= 0 \\ u_c[1 + Q_d b] + \beta E\{u'_c[\epsilon' \psi_d - (1-\lambda)\overline{R}(1+\lambda Q')]\} &= 0 \end{aligned}$$

- Where:

$$\begin{aligned} Q &= Q(a,b,d) \text{ as in auxiliary func } Q \text{ step 1} \\ Q' &= Q(a', b(\epsilon', a'), d(\epsilon', a')) \text{ as in auxiliary func } Q \text{ step 1} \\ Q_b &= Q_b(a,b,d) \text{ as in derivatives step 2: contains } d_a, b_a, Q'_b, Q'_d \\ Q_d &= Q_d(a,b,d) \text{ as in derivatives step 2: contains } d_\epsilon, b_\epsilon, Q'_b, Q'_d \\ Q'_b &= \mathcal{B}(\epsilon', a') \text{ from FOC in step 3} \\ Q'_d &= \mathcal{D}(\epsilon', a') \text{ from FOC in step 3} \end{aligned}$$

provided

$$\begin{aligned} \epsilon' &= \epsilon' \psi(d(\epsilon, a)) \\ a' &= \lambda a + b(\epsilon, a) + (1-\lambda)\overline{R} d(\epsilon, a) \\ b &= b(\epsilon, a) \\ d &= d(\epsilon, a) \end{aligned}$$

The indifference condition between both forms of moving resources

- When the solution is interior we can use both FOC to see what drives the indifference between both forms of "borrowing." Equating them and moving terms we get

$$u_c[(Q-1)+b\,(Q_b-Q_d)] = \beta E\{u_c'\,[\overline{R}(1-\lambda)-1]\,(1+\lambda Q')]\} - \beta E\{u_c'[\epsilon'\psi_d]\}$$

- The left hand side has the gains from borrowing versus not paying. We get directly that per unit that we borrow we get Q while if we default one unit we get the whole unit. The second consideration is the relative effect on the price of loans (to be analyzed below) multiplied by the amount of debt. Finally, tomorrow, both types of borrowing have differential rates of accumulation, (the difference being $[\overline{R}(1-\lambda)-1]$), and, defaulting has the lower subsequent output cost.

a decomposition of the Considerations

Recall that we wrote the GEE compactly as

$$\begin{aligned} u_c[Q + Q_b b] &= \beta E\{u_c'(1+\lambda Q')\} \\ u_c[1 + Q_d b] &= \beta E\{u_c'[(1-\lambda)\overline{R}(1+\lambda Q') - \epsilon' \psi_d]\} \end{aligned}$$

It interpretation is standard, with price derivatives Q_b and Q_d, for which we have explicit expressions, encapsulating all future consequences, including dilution. They can be written as

$$\begin{aligned} Q_b = \; & \frac{1}{1+r} E\left\{-(d'/a')_a \;\; \text{effect on \% defaulted via } a' \right. \\ & +\overline{R}(1-\lambda)(d'/a')_a Q \;\; \text{default that remains debt, via } a' \\ & \left. + \left(\lambda + \overline{R}(1-\lambda)\frac{d'}{a'}\right)[\mathcal{B}'(\lambda + b_a') + \lceil' d_a'] \right\} \;\; \text{dilution via } a'\text{; pf derivatives} \end{aligned}$$

$$\begin{aligned} Q_d = \; & \frac{1}{1+r} E\left\{-\frac{d_\epsilon' \epsilon' \psi_d}{a'} \;\; \text{effect on \% defaulted via } \epsilon' \right. \\ & +\overline{R}(1-\lambda)\frac{d_\epsilon' \epsilon' \psi_d}{a'} Q \;\; \text{default that remains debt, via } \epsilon' \\ & \left. + \left(\lambda + \overline{R}(1-\lambda)\frac{d'}{a'}\right)[\mathcal{B}' b_\epsilon' + \mathcal{D}' d_\epsilon']\epsilon' \psi_d \right\} \;\; \text{dilution via } \epsilon'\text{; pf derivatives} \\ & +(1-\lambda)\overline{R} Q_b \;\; \text{effect via } a'\text{; see } Q_b \text{ above} \end{aligned}$$

Writing them slightly differently

$$Q_b = \frac{1}{1+r} E\Big\{ -\frac{d_a(\epsilon',a')a' - d(\epsilon',a')}{a'^2}\Big[1+\overline{R}(1-\lambda)Q\Big] \text{ Increased default via } a'$$
$$+\Big(\lambda+\overline{R}(1-\lambda)\frac{d'}{a'}\Big)[\mathcal{B}'(\lambda+b'_a)+\mathcal{D}'d'_a] \text{ dilution via } a' \Big\}.$$

$$Q_d = \frac{1}{1+r} E\Bigg\{ \Big[1+\overline{R}(1-\lambda)Q\Big]\Bigg(-\frac{d'_\epsilon \epsilon' \psi_d}{a'} \quad \text{increased default via } \epsilon'$$
$$-(1-\lambda)\overline{R}\ \frac{d_a(\epsilon',a')a' - d(\epsilon',a')}{a'^2}\Bigg) \text{ and via } a'$$
$$+\Big(\lambda+\overline{R}(1-\lambda)\frac{d'}{a'}\Big)\ [\mathcal{B}'b'_\epsilon+\mathcal{D}'d'_\epsilon]\ \ \epsilon'\psi_d \text{ dilution via } \epsilon'$$
$$+[\mathcal{B}'(\lambda+b'_a)+\mathcal{D}'d'_a]\ (1-\lambda)\overline{R}\Big) \text{ and via } a' \Bigg\}.$$

Looking at the difference between Q_b and Q_d

$$Q_b - Q_d = \frac{1}{1+r} E\left\{ -(d'/a')_a \left[1 + \overline{R}(1-\lambda)Q\right] \right.$$
$$\left. + \left(\lambda + \overline{R}(1-\lambda)\frac{d'}{a'}\right)\left[\mathcal{B}'(\lambda + b'_a) + \lceil' d'_a\right] \right\} \left[1 - (1-\lambda)\overline{R}\,\right]$$

differences between b and d in effects on increased default and dilution via a'

$$+ \frac{1}{1+r} E\left\{ \left(\left[1 + \overline{R}(1-\lambda)Q\right] \frac{d'_\epsilon}{a'} - \left(\lambda + \overline{R}(1-\lambda)\frac{d'}{a'}\right) \left[\mathcal{B}' b'_\epsilon + \lceil' d'_\epsilon\right] \right) \epsilon' \psi_d \right\}.$$

only due to d, additional effects on increased default and dilution via ϵ'

The terms shaping dilution:

1. Working via a'

$$[\mathcal{B}'(\lambda + b'_a) + \mathcal{D}' d'_a] = \qquad (\lambda + b'_a)\left[\frac{\beta E\{u''_c(1+\lambda Q'')\} - Q' u'_c}{b' u'_c}\right] + \\ d'_a\left[\frac{-\beta E\{u''_c(\epsilon'' \psi'_d - (1-\lambda)\overline{R}(1+\lambda Q''))\} - u_c}{b' u'_c}\right]$$

- This is a weighted average of the time inconsistent elements associated to default and to save.

2. Working via ϵ'

$$[\mathcal{B}'\ b'_\epsilon + \mathcal{D}' d'_\epsilon] = \qquad b'_\epsilon\left[\frac{\beta E\{u''_c(1+\lambda Q'')\} - Q' u'_c}{b' u'_c}\right] + \\ d'_\epsilon\left[\frac{-\beta E\{u''_c(\epsilon'' \psi'_d - (1-\lambda)\overline{R}(1+\lambda Q''))\} - u_c}{b' u'_c}\right]$$

- This is also a weighted average of the time inconsistent elements associated to default and to save.

Conclusions

- We have developed a characterization of the equilibrium in a popular class of models widely used to treat issues of sovereign default. These models can be relatively sophisticated in terms of its ingredients.
- Such characterization looks at a problem of a decision maker that
  1. Takes as given the em decision rules of it future self.
  2. Faces market restrictions that can be dealt with as part of the problem.
- The characterization is in terms of functional equations where the terms involved have a clear economic interpretation and can be used to find the solution with arbitrary accuracy without constructing examples with desired properties by looking at a particular class of shocks.
- One of the equations involved is a GEE where the agent understands how future versions of itself will be affected by its current choices and tries to exploit them.

References

Adam, Klaus and Michael Grill. 2012. "Optimal Sovereign Default." CEPR Discussion Papers 9178, C.E.P.R. Discussion Papers. URL http://ideas.repec.org/p/cpr/ceprdp/9178.html.

Arellano, Cristina. 2008. "Default Risk and Income Fluctuations in Emerging Economies." *American Economic Review* 98 (3):690–712.

Arellano, Cristina, Xavier Mateos-Planas, and José-Víctor Ríos-Rull. 2013. "Partial Default." Unpublished Manuscript, University of Minnesota.

Arellano, Cristina and Ananth Ramanarayanan. 2012. "Default and the Maturity Structure in Sovereign Bonds." *Journal of Political Economy* 120 (2):187 – 232. URL http://EconPapers.repec.org/RePEc:ucp:jpolec:doi:10.1086/666589.

Chatterjee, S., D. Corbae, M. Nakajima, and J.-V. Ríos-Rull. 2007. "A Quantitative Theory of Unsecured Consumer Credit with Risk of Default." *Econometrica* 75 (6):1525–1589.

Eaton, Jonathan and Mark Gersovitz. 1981. "Debt with Potential Repudiation: Theoretical and Empirical Analysis." *Review of Economic Studies* 48 (2):289–309.

Gomes, Joao, Urban Jermann, and Lukas Schmid. 2014. "Sticky Leverage." Working Paper, University of Pennsylvania.

Klein, Paul, Per Krusell, and José-Víctor Ríos-Rull. 2008. "Time-Consistent Public Policy." *Review of Economic Studies* 75 (3):789–808.

Krusell, Per, Burhanettin Kuruşçu, and Anthony A. Smith. 2002. "Equilibrium Welfare and Government Policy with Quasi-Geometric Discounting." *Journal of Economic Theory* 105:42–72.

Livshits, Igor, James MacGee, and Michelle Tertilt. 2007. "Consumer Bankruptcy: A Fresh Start." *American Economic Review* 97 (1):402–418.

Passadore, Juan and Juan Xandri. 2015. "Robust Conditional Predictions in Dynamic Games: An Application to Sovereign Debt." Mimeo, MIT.

Perez, Diego. 2015. "Sovereign Debt, Domestic Banks and the Provision of Public Liquidity." Mimeo, Stanford University.

appendix

The long term debt problem

Properties of auxiliary functions

Given g and d, two of these are simple contractions:

- Prices

$$q(a'; g, d) = R^{-1}\left\{(1 - F(d(a'))) + (1-\lambda)\int_{d(a')} q(g(\epsilon', a'); g, d)\ f(d\epsilon')\right\}$$

- Continuation values associated with optimality (eg, continuity, FOC ...)

$$\widetilde{v}(\epsilon, a; g, d) = \begin{cases} \widehat{v}(\epsilon) & \text{if } \epsilon < d(a) \\ v(\epsilon, a; g, d) & \text{if } \epsilon \geq d(a) \end{cases}$$

with

$$v(\epsilon, a; g, d) = u(\epsilon + a + q(g(\epsilon, a); g, d)((1-\lambda)a - g(\epsilon, a))) + \beta \int \widetilde{v}(\epsilon', g(\epsilon, a); g, d) f(d\epsilon')$$

$$\widehat{v}(\epsilon) = u(\epsilon) + \frac{\beta}{1-\beta}\int u(\epsilon') f(d\epsilon')$$

definitions (3)

- Marginal utility, in future continuation allocations

$$\mathcal{U}_c(\epsilon, a; g, d) \equiv \mathrm{d}u(\epsilon + a + q(g(\epsilon, a); g, d)((1-\lambda)a - g(\epsilon, a)))/\mathrm{d}\ c$$

- Consumption from budget constraint

$$\mathcal{C}(\epsilon, a, a'; g, d) = \epsilon + a + q(a'; g, d)((1-\lambda)a - a')$$

Optimal default (4)

Threshold rule $\epsilon^*(a; g, d)$ is value ϵ^* solving

$$\widehat{v}(\epsilon^*) = u(\mathcal{C}(\epsilon, a, a'(\epsilon, a^*; g, d); g, d)) + \beta \int \widetilde{v}(a'(\epsilon, a^*; g, d), \epsilon'; g, d) f(d\epsilon') \tag{1}$$

where $a'(\epsilon, a; g, d)$ is optimal (deviation) choice of savings.

Optimal (deviation) savings: GEE (5)

The problem of the agent is

$$\max_{a'} u(\mathcal{C}(\epsilon, a, a'; g, d)) + \beta \int_0^{h(a')} \widehat{v}(\epsilon') f(d\epsilon') + \beta \int_{h(a')} v(\epsilon', a'; g, d) f(d\epsilon')$$

for $\epsilon > \epsilon^*(a; g, d)$, where optimal (deviation) default rule $\epsilon^*(a; g, d)$.

The FOC, envelope condition on v, and continuity imply the GEE

$$\begin{aligned} u_c(\mathcal{C}(\epsilon, a, a'; g, d))[q_a(a'; g, d)((1-\lambda)a - a') - q(a'; g, d)] \\ + \beta \int_{h(a')} \mathcal{U}_c(\epsilon', a'; g, d)[1 + (1-\lambda) q(g(\epsilon', a'); g, d)] f(d\epsilon') \end{aligned} \quad (2)$$

which yields the savings rule $a'(\epsilon, a; g, d)$.

Notice this involves the derivative of the price $q_a(a'; g, d)$.

The derivative of the price (6)

Differentiating the expression for q, shows that the derivative $q_a(a'; g, d)$ depends on the derivative of future prices.

The FOC holds in future from which future derivatives $q_a(g(\epsilon', a'); g, d)$ can be expressed as $\dashv(\epsilon', a'; g, d)$ where

$$\begin{aligned} \dashv(\epsilon, a; g, d) \equiv \{ & q(g(\epsilon, a); g, d)\mathcal{U}_c(\epsilon, a; g, d) \\ & - \beta \int_{h(g(\epsilon,a))} \mathcal{U}_c(\epsilon', g(\epsilon, a); g, d) \\ & \times [1 + (1-\lambda)\, q(g(\epsilon', g(\epsilon, a)); g, h)] f(d\epsilon') \\ & \} \\ & / \{[(1-\lambda)a - g(\epsilon, a)]\mathcal{U}_c(\epsilon, a; g, h)\} \end{aligned}$$

Then differentiating the auxiliary equation for q gives

$$\begin{aligned} q_a(a'; g, h) = R^{-1}[& -F_h(h(a'))h_a(a') + (1-\lambda) \\ & \{-h_a(a')q(g(h(a'), a'); g; h) f(h(a')) \\ & + \int_{h(a')} \dashv(\epsilon', a'; g, h) g_a(\epsilon', a') f(d\epsilon')\}] \end{aligned} \qquad (3)$$

Equilibrium and computation (7 and 8)

An equilibrium is a pair of functions g and d such that:

1. Fixed point. Optimal choices in (1) and (2)+(3) are consistent with g and d:

$$\begin{aligned} a'(\epsilon, a; h, g) &= g(\epsilon, a) \\ \epsilon^*(a; g, h) &= h(a) \end{aligned}$$

2. Given g and d, the underlying auxiliary functions are determined as in (69) and (69).

There are two loops, the second is the outer loop. Two possible approaches to inner loop: solve as fixed point iterating on g and d; or solve as a system of equations in g and d. In the second approach, we could write the system compactly:

$$\begin{aligned} E\xi^{GEE}(\epsilon, a, h(g((\epsilon, a))), h_a(g(\epsilon, a)), g(\epsilon, a), g(\epsilon', g(\epsilon, a)), g_a(\epsilon', g(\epsilon, a)), \epsilon') &= 0 \\ E\xi^{H}(a, h(a), g(a, h(a))) &= 0 \end{aligned}$$