Macro Het Agents 081

Preliminary

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Measure Theory

Measure theory is a tool that helps us aggregate.

Definition
For a set S, $\mathcal{S}$ is a family of subsets of S, if $B \in \mathcal{S}$ implies $B \subseteq S$ (but not the other way around).

Remark
Note that in this section we will assume the following convention

1. *small letters (e.g. s) are for elements,*
2. *capital letters (e.g. S) are for sets, and*
3. *fancy letters (e.g. $\mathcal{S}$) are for a set of subsets (or families of subsets).*

Definition

A family of subsets of S, $\mathcal{S}$, is called a σ-algebra in S if

1. $S, \emptyset \in \mathcal{S}$;
2. if $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to complements and $A^c = S \backslash A$); and,
3. for $\{B_i\}_{i \in \mathbb{N}}$, if $B_i \in \mathcal{S}$ for all $i \Rightarrow \bigcap_{i \in \mathbb{N}} B_i \in \mathcal{S}$ (i.e. $\mathcal{S}$ is closed with respect to countable intersections.

Example

1. The power set of S and $\{\emptyset, S\}$ are σ-algebras in S.
2. $\{\emptyset, S, S_{1/2}, S_{2/2}\}$, where $S_{1/2}$ means the lower half of S (imagine S as an closed interval in $\mathbb{R}$), is a σ-algebra in S.
3. If $S = [0, 1]$, then $\mathcal{S} = \left\{\emptyset, \left[0, \frac{1}{2}\right), \left\{\frac{1}{2}\right\}, \left[\frac{1}{2}, 1\right], S\right\}$ is *not* a σ-algebra in S. But $\mathcal{S} = \left\{\emptyset, \left\{\frac{1}{2}\right\}, \left\{\left[0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right]\right\}, S\right\}$ is.

It allows us to define sets where things happen and we can *weigh* those sets (avoiding math troubles)

Definition

Suppose $\mathcal{S}$ is a σ-algebra in S. A measure is a real-valued function $x : \mathcal{S} \to \mathbb{R}_+$, that satisfies

1. $x(\emptyset) = 0$;
2. if $B_1, B_2 \in \mathcal{S}$ and $B_1 \cap B_2 = \emptyset \Rightarrow x(B_1 \cup B_2) = x(B_1) + x(B_2)$ (additivity); and,
3. if $\{B_i\}_{i \in \mathbb{N}} \in \mathcal{S}$ and $B_i \cap B_j = \emptyset$ for all $i \neq j \Rightarrow x(\cup_i B_i) = \sum_i x(B_i)$ (countable additivity).

A set S, a σ-algebra in it ($\mathcal{S}$), and a measure on $\mathcal{S}$ x, define a measurable space, $(S, \mathcal{S}, x)$.

Definition
A Borel σ-algebra is a σ-algebra generated by the family of all open sets $\mathfrak{B}$ (generated by a topology). A Borel set is any set in $\mathfrak{B}$.

A Borel σ-algebra corresponds to complete information.

Definition
A probability measure is measure where $x(S) = 1$. $(S, \mathcal{S}, x)$ is a probab space. The probab of an event is then given by $x(A)$, where $A \in \mathcal{S}$.

Definition
Given a m'able space $(S, \mathcal{S}, x)$, a real-valued function $f : S \to \mathbb{R}$ is m'able (with respect to the m'able space) if, for all $a \in \mathbb{R}$, we have

$$\{b \in S \mid f(b) \leq a\} \in \mathcal{S}.$$

Interpret σ-algebras as describing available information.

Similarly, a function is m'able wrt a σ-algebra $\mathcal{S}$, if it can be evaluated

Example

Suppose $S = \{1, 2, 3, 4, 5, 6\}$. Consider a function f that maps the element 6 to the number 1 (i.e. $f(6) = 1$) and any other elements to -100. Then f is NOT measurable with respect to $\mathcal{S} = \{\emptyset, \{1, 2, 3\}, \{4, 5, 6\}, S\}$. Why? Consider $a = 0$, then $\{b \in S \mid f(b) \leq a\} = \{1, 2, 3, 4, 5\}$. But this set is not in $\mathcal{S}$.

Extend the notion of Markov stuff to any measurable space

Definition

Given a measurable space $(S, \mathcal{S}, x)$, a function $Q : S \times \mathcal{S} \to [0, 1]$ is a transition probability if

1. $Q(s, \cdot)$ is a probability measure for all $s \in S$; and,
2. $Q(\cdot, B)$ is a measurable function for all $B \in \mathcal{S}$.

Intuitively, for $B \in \mathcal{S}$ and $s \in S$, $Q(s, B)$ gives the probability of being in set B tomorrow, given that the state is s today.

1. A *Markov chain* with transition matrix given by

$$\Gamma = \begin{bmatrix} 0.2 & 0.2 & 0.6 \\ 0.1 & 0.1 & 0.8 \\ 0.3 & 0.5 & 0.2 \end{bmatrix},$$

on $S = \{1, 2, 3\}$, with the the power set being the σ-algebra of S).

$$Q(3, \{1, 2\}) = \Gamma_{31} + \Gamma_{32} = 0.3 + 0.5 \ .$$

2. Consider a measure x on $\mathcal{S}$. x_i is the fraction of type i. Then

$$\begin{aligned} x_1' &= x_1\Gamma_{11} + x_2\Gamma_{21} + x_3\Gamma_{31}, \\ x_2' &= x_1\Gamma_{12} + x_2\Gamma_{22} + x_3\Gamma_{32}, \\ x_3' &= x_1\Gamma_{13} + x_2\Gamma_{23} + x_3\Gamma_{33}. \end{aligned}$$

In other words: $\mathrm{x}' = \Gamma^T \mathrm{x}$, where $\mathrm{x}^T = (x_1, x_2, x_3)$.

From a measure x today to one tomorrow x'

$$\begin{aligned} x'(B) &= T(x, Q)(B) \\ &= \int_S Q(s, B)\, x(ds), \quad \forall B \in \mathcal{S}, \end{aligned}$$

we integrated over all $s \in S$ to get the prob of B tomorrow.

A stationary distribution is a fixed point of T, that is x^* such that

$$x^*(B) = T(x^*, Q)(B), \quad \forall B \in \mathcal{S}.$$

Theorem

If Q has nice properties (American Dream and Nightmare) then $\exists$ a unique stationary distribution x^* and

$$x^* = \lim_{n\to\infty} T^n(x_0, Q), \qquad \textit{for any } x_0.$$

Exercise

Consider unemployment in a very simple economy (in which the transition matrix is exogenous). There are two states of the world: being employed and being unemployed. The transition matrix is given by

$$\Gamma = \begin{pmatrix} 0.95 & 0.05 \\ 0.50 & 0.50 \end{pmatrix}.$$

Compute the stationary distribution corresponding to this Markov transition matrix.

Industry Equilibrium

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- n^* is an increasing function of both arguments. Prove it.

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- x is a measure on $(S, \mathcal{S})$, which describes the cross-sectional distribution of productivity among firms.
- Use x to define statistics of the industry: Since individual supply is $sf\left(n^*\left(s,p\right)\right)$, then the aggregate supply

$$Y^S\left(p\right) = \int_S sf\left(n^*\left(s,p\right)\right) x\left(ds\right). \tag{3}$$

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- Let Demand $Y^D\left(p\right)$. Then p^* clears the market:

$$Y^D\left(p^*\right) = Y^S\left(p^*\right). \tag{4}$$

Where does x come from?

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- The choice is static. The value of an s firm is

$$V(s;p) = \sum_{t=0}^{\infty} \left(\frac{\delta}{1+r}\right)^{t} \pi(s,p) = \qquad \left(\frac{1+r}{1+r-\delta}\right) \pi(s,p)$$

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- Entrants draw s from probability measure γ over $(S, \mathcal{S})$.

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- Assume a fixed entry cost, c^E before learning s. Value of an entrant

$$V^E(p) = \int_S V(s;p)\,\gamma(ds) - c^E. \tag{5}$$

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- Equilibrium requires $V^E = 0$

- x_t : cross-sectional distribution of firms. For any $B \subset S$, fraction $1-\delta$ of firms with $s \in B$ die and mass m of newcomers enter. Next period's measure of firms on set B is

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- Cross-sectional distribution of firms completely determined by γ.
- Consider an updating operator T

$$Tx(B) = \delta x(B) + m\gamma(B), \quad \forall B \in \mathcal{S}, \tag{7}$$

a stationary dbon is a fixed point, i.e. x^* such that $Tx^* = x^*$, so

$$x^*(B; m) = \frac{m}{1-\delta}\gamma(B), \quad \forall B \in \mathcal{S}. \tag{8}$$

- Demand and supply condition in equation (4) is

$$Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right), \tag{9}$$

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Definition

A stationary distribution for this environment consists of functions V, π^*, n^*, p^*, x^*, and m^*, that satisfy:

1. Given prices, V, π^*, and n^* solve the incumbent firm's problem;
2. $Y^D\left(p^*\left(m\right)\right) = \int_S s\ f\left[n^*\left(s;p\right)\right]\ dx^*\left(s;m\right)$;
3. $\int_s V\left(s;p\right)\gamma\left(ds\right) - c^E = 0$; and,
4. $x^*\left(B\right) = \delta x^*\left(B\right) + m^*\gamma\left(B\right), \quad \forall B \in \mathcal{S}$.

- Assume s follows a Markov process with transition Γ. This would change the mapping T in Equation (7) to

$$Tx(B) = \delta \int_S \Gamma(s, B)\, x(ds) + m\gamma(B), \quad \forall B \in \mathcal{S}. \tag{10}$$

But no firm exits (c^E is a sunk cost). Still not much Econ.

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  - But it is not enough. Assume Γ satisfies stochastic dominance: $s^1 > s^2$ implies $\sum_{s'=1}^{\widehat{s}} \Gamma_{s^1,s'} < \sum_{s'=1}^{\widehat{s}} \Gamma_{s^2,s'}$.

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  - Then $\exists$ a threshold, $s^* \in S$, below which firms exit and above stay.

$$V(s; p) = \max \left\{ 0, \pi(s; p) + \frac{1}{(1+r)} \int_S V(s'; p)\, \Gamma(s, ds') - c^v \right\}. \tag{11}$$

- Updating operator becomes

$$x'(B) = \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x(ds) + m\gamma(B \cap [s^*, \bar{s}]), \quad \forall B \in \mathcal{S}. \tag{12}$$

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  - Threshold for being in top 10% by size? Solve for $\widehat{s}$

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$$\frac{\int_{\hat{s}}^{\bar{s}} x^*(ds)}{\int_{s^*}^{\bar{s}} x^*(ds)} = 0.1,$$

  - Fraction of workers in largest top 10% of firms

$$\frac{\int_{\hat{s}}^{\bar{s}} n^*(s,p)\, x^*(ds)}{\int_{s^*}^{\bar{s}} n^*(s,p)\, x^*(ds)}.$$

Do

Exercise

Compute the average growth rate of the smallest one third of the firms.

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What would be the fraction of firms in the top 10% largest firms in the economy that remain in the top 10% in next period?

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3. Free-entry condition: $V^E(p^*) = 0.$
4. For any $B \in \mathcal{S}$

$$x^*(B) = m^* \gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}]) x^*(ds)$$

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2. The reservation productivity s^* satisfies $d^*(s; p^*) = \begin{cases} 1 & \text{if } s \geq s^* \\ 0 & \text{otherwise} \end{cases}$.
3. Free-entry condition: $V^E(p^*) = 0.$
4. For any $B \in \mathcal{S}$

$$x^*(B) = m^*\gamma(B \cap [s^*, \bar{s}]) + \int_{s^*}^{\bar{s}} \Gamma(s, B \cap [s^*, \bar{s}])\, x^*(ds)$$

5. Market clearing:

$$Y^d(p^\star) = \int_{s^\star}^{\bar{s}} s\ f(n^\star(s; p^\star))x^\star(ds)$$

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$$\forall \, B^{S} \in \mathcal{S}, \forall \, B^{N} \in \mathcal{N}.$$

Exercises

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- Consider demand shifters z_t so that $D(P, z_t)$ where $z_{t+1} = \phi(z_t)$ so we can choose to represent it as a sequence or recursively.

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- Then do linear approximations in sequence space.

- Consider the social planner's problem (with full depreciation)

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\begin{aligned}
V(k_t) &= \max_{c_t, k_{t+1}} u(c_t) + \beta V(k_{t+1}) \\
&\text{s.t. } c_t + k_{t+1} \leq f(k_t), \qquad \forall\ t \geq 0 \\
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u_c(c_t) = \beta\ u_c(c_{t+1})\ f_k(k_{t+1}), \forall\ t \geq 0
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a second order difference equation with exactly two boundary conditions, k_0 and $k_\infty = k^*$.

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- Look at the a steady state k^*
- Rewrite solution as

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- This is in fact an impulse response function.

- We want now to simulate a response of the economy to shocks. Consider an AR(1) process for z_t : with $z_{t+1} = \rho^t z_t + \epsilon_{t+1}$.) where $\epsilon_t \sim \mathcal{N}(\int, \supset^{\in})$.

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Log- Linear Approximation in the Simplest Growth Model II

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&\vdots \\
\widetilde{k}_{t+1}(k_0, \epsilon^t) &= \sum_{\tau=0}^{t} \epsilon_t \ \widehat{k}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = 1, \ \epsilon_t = 0, \forall t \neq 0,
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5. *Imagine now that the industry is subject to demand shocks that follow an AR(1). Describe an algorithm to approximate it.*

Incomplete Market Models

- Consider the problem of a farmer with storage possibilities

$$V(s,a) = \max_{c,a' \geq 0} \quad u(c) + \beta \sum_{s'} \Gamma_{ss'} \; V(s', a') \qquad s.t.$$

$$c + qa' = a + s$$

a assets, c consumption, and $s \in \{s^1, \cdots, s^{N^s}\} = S$ has transition Γ. q units today yield 1 unit tomorrow. Only nonnegative storage.

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- For any such prob measure x its evolution is

$$x'\left(B\right) = \widetilde{T}(B,x;\Gamma,g) = \sum_{s}\int_{0}^{\bar{a}}\sum_{s'\in B_s}\Gamma_{ss'}\;1_{\{g(s,a)\in B_a\}}\;x\left(s,da\right),\quad \forall B\in\mathcal{B}$$

where B_s and B_a are projections of B on S and A,

Theorem

With a well behaved Γ, there is a unique stationary probability x^*, so that:

$$x^*(B) = \widetilde{T}(B, x^*; \Gamma, g)(B), \quad \forall B \in \mathcal{B},$$
$$x^*(B) = \lim_{n \to \infty} \widetilde{T}^n(B, x_0; \Gamma, g)(B), \quad \forall B \in \mathcal{B},$$

for all initial probability measures X_0 on $(E, \mathcal{B})$.

We use compactness of $[0, \overline{A}]$.

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- Or it could be tighter which makes it likely to bind $0 > \underline{a} > a^n$.

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  3. $\lim_{q\to\infty} \int_{A\times S} a\, dX^*(q) < 0$. As $q \to \infty$, arbitrary large consumption is achievable by borrowing.

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- Now we need $\beta(1+r) < 1$. We write

$$V(s,a) = \max_{c,a'\geq 0} \quad u(c) + \beta \int_{s'} V\left(s',a'\right) \Gamma(s, d\,s') \qquad s.t.$$
$$c + a' = (1+r)\,a + ws$$

where r is the return on savings and w is the wage rate.

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Rewrite the economy when households like leisure

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- Otherwise the best tax policy in the Rep agent (which is Pareto Optimal) would be to subsidize capital to maximize steady state consumption.

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- Computationally, this problem is a beast! So, what then?

- They people believe tomorrow's capital depends only on K and not on x. This, obviously, is not an economy with rational expectations. The agent's problem in such a setting is

$$\begin{aligned}\widetilde{V}(z,K,s,a) = \max_{c,a'} \quad & u(c) + \beta \sum_{z',s'} \Pi_{zz'} \Gamma_{ss'}^{z'} \widetilde{V}\left(z',K',s',a'\right) \\ s.t. \quad & c + a' = a\ z\ f_k\left(K,\bar{N}\right) + szf_n\left(K,\bar{N}\right) \\ & K' = \widetilde{G}(z,K)\end{aligned}$$

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- They found it works well in boring settings (things are pretty linear)

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- Valuable for SMALL shocks like all linear approximations.

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    - We look for a fixed point of this (not necessarily iterating mechanically but as solution of a system of equations)

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&\vdots \\
\widetilde{d}_{t+1}(x_0, \epsilon^t) &= \sum_{\tau=0}^{t} \frac{\epsilon_t}{\bar{\epsilon}_0}\ \widehat{d}_{t-\tau+1} \qquad \text{exact if } \epsilon_0 = \widetilde{\epsilon}_0,\ \epsilon_t = 0, \forall t \neq 0.
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- s is Markovian (Γ) labor labor productivity. Then the unemployed

$$V(s, 0, a) = \max_{c,h,a' \geq 0} u(c, h) + \beta \sum_{s'} \Gamma_{ss'} \left[\phi(h) V(s', 1, a') + (1 - \phi(h)) V(s', 0, a') \right]$$

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$$V(s, 1, a) = \max_{c,a' \geq 0} u(c) + \beta \sum_{s'} \Gamma_{ss'} \left[\delta V(s', 0, a') + (1 - \delta) V(s', 1, a') \right]$$

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- If worker

$$V^w(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[dV^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

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- Similarly, the entrepreneur's problem can be formulated as follows

$$V^e(s,\eta,a) = \max_{c,a'\geq 0, d\in\{0,1\}} u(c) + \beta \sum_{s',\eta'} \Gamma_{ss'}\Gamma_{\eta\eta'} \left[d \; V^w(s',\eta',a') + (1-d)V^e(s',\eta',a')\right]$$

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- Income is from profits $\pi(a,s,\eta)$ not wages. Assume entrepreneurs have a DRS technology f. Profits are

$$\pi(s,\eta,a) = \max_{k,n} \eta f(k,n) + (1-\delta)k - (1+r)(k-a) - w\max\{n-s,0\}$$

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- The constraint here reflects the fact that entrepreneurs can only make loans up to a fraction ϕ of his total wealth.

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- With financial constraints wealth matters. Wealthy agents with high h will while the poor with low η will not.
- For the rest, it depends. If η is persistent, poor individuals with high entrepreneurial ability will save to one day become entrepreneurs, while rich agents with low entrepreneurial ability will lend their assets and become workers.

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- What determines $q(a')$? A zero profit on lenders: Probability of default

Agents in Aiyagari worlds with Extreme Value Shocks

- The fundamental problem

$$v(s,a) = \max_{a', c = sw + aR - a'} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \epsilon(c) + \sum_{s'} \Gamma_{s,s'} \; v(s', a') \right\}$$

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- Fix N, a large integer, we approximate the problem by

$$v(s,a) = \max_{a^{n\prime} = sw + aR - c^n, c^n} \left\{ \frac{c^{1-\sigma} - 1}{1-\sigma} + \epsilon^n + \sum_{s'} \Gamma_{s,s'} \; v(s', a^{n\prime}) \right\}$$

We have to impute the right probabilities

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- We may just want to be realistic about the finite nature of the length of life.

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- Consider

$$m_t = \frac{\omega^y - c_t^y}{p_t}$$

$$c_{t+1}^o + = \frac{m_t}{p_{t_+1} + m_t}$$

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- Still, Why accept Money from older agents? Who needs them?

Growth Model with Many Firms Suitable for Pandemic Times

- This is a growth model suitable to study business cycles.
- Emphasis on small business creation not on inequality so rep hholds.
- Creation and destruction of small firms both for technological and for financial reasons.
- Household cannot help its small businesses in distress.
- We have in mind that even though Pandemic affects both Supply (want less work) and Demand (Less consumption) there is a reduction in output sold per unit of good produced of $\phi(S)$.

- Two sectors as in Quadrini (2000): Corporate and non corporate sector.
- Corporate sector uses capital and labor via aggr prod fn $F(K, N)$
- Non corporate sector: type/size firms $i \in \{1, \cdots, I\}$, $f^i(n)$, $f^i_n > 0$, (provided the firm has the required number of managers, λ^i).
- A firm requires creation: It costs ξ^i to open a new firm of size i.
- Some Firms are destroyed.
  - Firms invest m in maintenance.
  - Probability that a firm survives is $q^i(m)$, $q^i(0) = 0$, $q^i(\infty) < 1$, $q^i_m > 0$.
- Aggregate measure of type i firms is X_i
- The law of motion of new firms is:

$$X_i' = q^i(M_i)\ X_i + B_i$$

- The Aggregate Feasibility Constraint is

$$C + [K' - (1-\delta)K] + \sum_i X_i\ M_i + \sum_i B_i\ \xi_i = \sum_i X_i\ f_i(N_i) + F(K, N).$$

- Household owns measure x_i of firms of type $i \in \{1, \cdots, \mathcal{I}\}$
- The household may be rationed in its workforce: i.e. it may not be in its static Euler equation.
- Households create b^i new firms of type i at cost ξ^i each,
- Managers choose maintenance and profits.
- In addition to its firms, households own a units of corporate capital which they can increase by savings.
- Households allocate its members to managers, workers or enjoyers of leisure:

$$n + \sum_i \lambda^i \, x^i + \ell = 1.$$

(implicitly we are guessing (to be verified) that all business are operated).

- Households have preferences over consumption c and leisure ℓ, using utility function $u(c, \ell)$ and discounts the future at rate β.

- Small firms cannot access financing once they are born.
- They can only give benefits to the household:

$$\Omega^i(S) = \max_{n \geq 0, m \leq \psi(S) f^i(n) - w\ n} \psi(S)\ f^i(n) - w\ n - m + \frac{q^i(m)}{R(S')} \Omega^i(S')$$

Here, S is the aggregate state and s in the individual state, $\Psi(S) < 1$ is capacity used which is demand determined and $R(S')$ is the rate of return used by the firm.

- Implictly assuming that there is no need to index $\Omega^i(S)$ by s.

Exercise

Get the FOC assuming first that m is unrestricted and then that $m \leq \psi(S) f^i(n) - w\ n$.

$$V(S,a,x_1,\cdots,x_I) = \max_{c,n,b_1,\cdots,b_I,a'} \; u(c,\; 1-n-\sum_i \lambda^i\; x^i) + \beta\; V(S',a',x_1',\cdots,x_I') \qquad s.t.$$

$$c + \sum_i \; b_i\; \xi_i + a' = n\; w(S) + a\; R(S) + \sum_i \; \pi_i(S)\; x_i$$

$$x_i' = q^i(M_i)\; x_i + b_i \qquad i \in \{1,\cdots,I\}.$$

Exercise

Get the FOCs for b^i a' and n assuming first that $\lambda^i = 0$ and $\pi^i > 0$ and charaterize the solution (the relation between the FOC of b^i, m^i and a'). Then characterize the FOC when $\lambda^i > 0$.

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