Convergence Proof of the Velocity Field for a Stokes Flow Immersed Boundary Method

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Abstract

The immersed boundary (IB) method is a computational framework for problems involving the interaction of a fluid and immersed elastic structures. It is characterized by the use of a uniform Cartesian mesh for the fluid, a Lagrangian curvilinear mesh on the elastic material, and discrete delta functions for communication between the two grids. We consider a simple IB problem in a two-dimensional periodic fluid domain with a one-dimensional force generator. We obtain error estimates for the velocity field of the IB solution for the stationary Stokes problem. We use this result to prove convergence of a simple small-amplitude dynamic problem. We test our error estimates against computational experiments. © 2007 Wiley Periodicals, Inc.

1 Introduction

The immersed boundary method was first introduced by Peskin in [24] as a computational tool to investigate blood flow in the presence of cardiac valves. It has since proved to be a generally useful computational framework for solving problems in fluid-structure interaction. The method has found many applications, including blood flow in the heart [17], vibrations of the cochlear basilar membrane [1, 10], blood clotting [7, 39, 42], aquatic locomotion [4, 6, 8, 14], insect flight [18, 19], flow with suspended particles [9, 33], and other physical problems [5, 15, 16, 29]. We refer to [25] for a more extensive list of applications.

In this paper, we present a proof of convergence for a stationary immersed boundary problem and a small-amplitude dynamic problem. We shall be concerned solely with the velocity field. We consider a model problem that consists of a one-dimensional closed filament Γ in a two-dimensional periodic fluid domain $\mathbb{U}$. This is a standard model problem for the immersed boundary method. A Lagrangian coordinate θ is placed on the one-dimensional filament so that $\mathbf{X}(\theta)$ traces the curve Γ. A boundary force $\mathbf{F}$ is given as a function of θ. Our problem is to find the fluid velocity field $\mathbf{u}$ under Stokes flow.

The problem can be formulated as an interface problem, where the Stokes equations are satisfied in the two regions demarcated by Γ. One requires that the velocity field $\mathbf{u}$ be continuous at Γ and satisfy stress jump conditions expressed in terms of $\mathbf{F}$ and $\mathbf{X}$. A mathematically equivalent way to formulate this problem is

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to consider Stokes flow under a singular external force field $\mathbf{f}$ [26]. This force field $\mathbf{f}$ is supported on the curve Γ and is expressed in terms of the boundary force $\mathbf{F}$ as follows:

$$\mathbf{f} = \int \mathbf{F}(\theta)\delta(\mathbf{x} - \mathbf{X}(\theta))d\theta \tag{1.1}$$

where $\mathbf{x}$ is a point in the fluid domain and δ is the Dirac delta function. The immersed boundary method uses this formulation to approach the problem.

We place a uniform Cartesian mesh on the fluid domain. In order for the singular force field $\mathbf{f}$ to be "felt" by the Cartesian grid, we must regularize $\mathbf{f}$. This is achieved by making use of discrete delta functions, whose properties will be discussed in Section 3. In this paper, we shall employ a spectral discretization for the periodic Stokes problem.

Our problem is thus to solve a partial differential equation with a singular source term. Such problems arise not only in the context of the immersed boundary method but also in the front tracking method [37] and the level set method [23, 30]. The numerical approximation of such problems has been analyzed in detail for the one-dimensional Poisson problem in [2, 36, 38], and a general framework for a convergence analysis of such problems is proposed in [36]. However, to the best of the author's knowledge, there has been no convergence analysis for such problems for PDEs in dimension 2 or higher.

In Section 2, we discuss the model problem above and its immersed boundary discretization. Following the framework proposed in [36], we write the exact velocity field and its immersed boundary approximation in the form of an integral and a sum, respectively. This reduces the problem to that of an approximation of an integral. In Section 3, we review some properties of the discrete delta functions given in [2, 32, 36] and present them in a fashion useful for our analysis.

In Section 4, we prove some estimates on the discrete and continuous Green's functions. A key estimate is Proposition 4.5 on the difference between the discrete and continuous Green's functions. We obtain this estimate by studying their Fourier sum representations.

In Section 5, we prove pointwise error estimates valid away from the immersed boundary Γ. Let the fluid mesh spacing be h. Suppose we refine the Lagrangian mesh so that the mesh spacing $\Delta\theta$ is proportional to h^α, $\alpha > 0$. If $\mathbf{F}$ and $\mathbf{X}$ are smooth functions of θ, we shall see that the following error estimate holds for "generic" (the precise meaning of which will be discussed in Section 5) points $\mathbf{x}$ away from Γ:

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \leq C\left(h(h^\alpha + h\log(h^{-1})) + h^p\right) \tag{1.2}$$

where $\mathbf{u}_h$ denotes the immersed boundary solution and p is the number of discrete moment conditions satisfied by the discrete delta function (see Section 3). If this estimate is optimal, it shows that $\alpha = 1$ and $p = 2$ are the optimal choices for these parameters.

In Section 6, we prove a global L^∞ error estimate for the velocity field of the immersed boundary solution. This estimate ensures that the velocity field converges to the true solution up to the immersed boundary. Suppose we refine $\Delta\theta$ proportionally to h^α. We shall prove

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty(\mathbb{U})} \le C(h+h^\alpha)\log(h^{-1}). \tag{1.3}$$

This shows that the immersed boundary solution converges to the true solution everywhere as long as $\alpha > 0$. This also shows that $\Delta\theta$ should be refined proportionally to h to obtain the optimal global convergence rate.

The stationary immersed boundary problem considered above constitutes only a part of a full immersed boundary computation, which may be described algorithmically as follows:

(1) Compute the boundary force $\mathbf{F}(\theta)$ given the configuration $\mathbf{X}(\theta)$.
(2) Given $\mathbf{F}(\theta)$, compute the regularized external force field $\mathbf{f}$ using discrete delta functions.
(3) Compute the velocity field $\mathbf{u}$.
(4) Interpolate the velocity field $\mathbf{u}$ at the immersed boundary points.
(5) Advect the immersed boundary points $\mathbf{X}(\theta)$ using the interpolated velocities.

So far we have only dealt with steps 2 and 3 above. In Section 7, we prove convergence of a simple dynamic immersed boundary problem. We consider an approximate problem in which we assume that the immersed boundary displacements are small enough so that the spreading (step 2) and the interpolation (step 4) operations may be performed at fixed locations. We take a Hookean restoring force to the fixed locations as the force law used in step 1. The global L^∞ estimate proven in Section 6 allows us to perform step 4, which involves point evaluation of the velocity field. This estimate also gives us the requisite consistency.

In Section 8, we shall test the error estimates for the stationary problem against a computational experiment for various choices of the discrete delta function. All computational results are consistent with the error estimates. We shall see that the error estimates capture the overall features of the convergence rates.

2 Model Problem

We consider the following force balance problem for Stokes flow (Figure 2.1). The equations we solve are:

$$\Delta\mathbf{u} = \nabla p - \mathbf{f} + \mathbf{g}, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\mathbf{f} = \int_{-\pi}^{\pi} \mathbf{F}(\theta)\delta(\mathbf{x}-\mathbf{X}(\theta))\,d\theta. \tag{2.3}$$

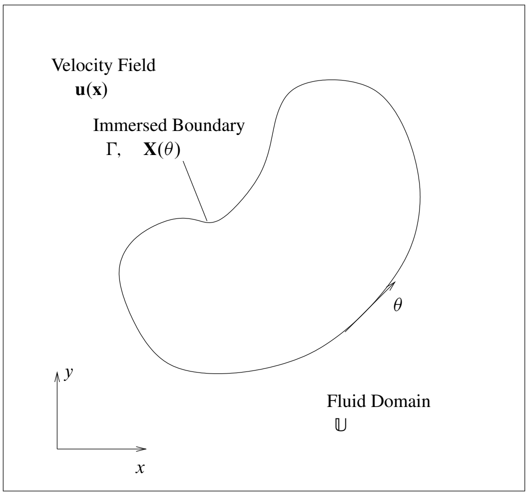

FIGURE 2.1. Schematic diagram describing the model problem.

The equations have already been rendered dimensionless. We shall work on a square periodic fluid domain $\mathbf{x} = (x, y) \in \mathbb{U}$ where $-\pi \leq x \leq \pi$ and $-\pi \leq y \leq \pi$. The immersed boundary $\mathbf{X}(\theta)$ parametrized by $\theta \in \Theta \equiv \mathbb{R}/2\pi\mathbb{Z}$ defines a simple closed curve Γ in the fluid domain. We note that θ is not necessarily an angular coordinate. We require that Γ stay away from $\partial\mathbb{U}$, where $\partial\mathbb{U}$ consists of the points $\mathbf{x} \in \mathbb{U}$ that satisfy $x = \pm\pi$ or $y = \pm\pi$. The force $\mathbf{F}(\theta)$ is a given function of the immersed boundary coordinate θ.

We have an additional constant force term $\mathbf{g}$ in the force balance equation (2.1) to ensure that the above problem has a solution in a periodic domain:

$$\begin{aligned} \int_{\mathbb{U}} (\mathbf{f} - \mathbf{g}) d\mathbf{x} &= \int_{\mathbb{U}} (-\Delta \mathbf{u} + \nabla p) d\mathbf{x} \\ &= \int_{\partial\mathbb{U}} (-\nabla \mathbf{u} + p)\mathbf{n}\, dS = 0 \end{aligned} \tag{2.4}$$

where $\mathbf{n}$ is the unit normal on $\partial\mathbb{U}$ and dS signifies integration on $\partial\mathbb{U}$. The last equality holds because $\mathbf{u}$ and p are assumed periodic. Using (2.3), we conclude that

$$\mathbf{g} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \mathbf{F}(\theta) d\theta. \tag{2.5}$$

The force field $\mathbf{g}$ serves to subtract the uniform (zero wavelength) component of the boundary force field. We must also place a constraint on the velocity field $\mathbf{u}$ so that we have a unique solution. We require that

$$\int_{\mathbb{U}} \mathbf{u}\, d\mathbf{x} = 0. \tag{2.6}$$

In what follows, we assume that $\mathbf{u}$ satisfies this constraint.

We now discretize the above problem using the immersed boundary method:

$$L_h \mathbf{u}_h = \mathbf{D}_h p_h - \mathbf{f}_h + \mathbf{g}_h, \tag{2.7}$$

$$\mathbf{D}_h \cdot \mathbf{u}_h = 0, \tag{2.8}$$

$$\mathbf{f}_h = \sum_{m=1}^{M} \mathbf{F}(\theta_m)\delta_h(\mathbf{x} - \mathbf{X}(\theta_m))\Delta\theta. \tag{2.9}$$

In the above, h is the mesh width of the uniformly discretized fluid domain and $\Delta\theta$ is the width of the immersed boundary mesh. We let

$$Nh = 2\pi, \tag{2.10}$$

$$M\Delta\theta = 2\pi, \quad \theta_m = -\pi + m\Delta\theta. \tag{2.11}$$

The discrete operators L_h and $\mathbf{D}_h$ are approximations to the Laplacian and the gradient/divergence operators, respectively. The function $\delta_h(\mathbf{x} - \mathbf{X})$ in (2.9) is a discrete approximation to the Dirac delta function centered around $\mathbf{X}$. The discrete delta function δ_h will be discussed in detail in Section 3.

We must supplement the above with an expression for $\mathbf{g}$ furnished here as a simple discretization of (2.5):

$$\mathbf{g}_h = \frac{1}{(2\pi)^2} \sum_{m=1}^{M} \mathbf{F}(\theta_m)\Delta\theta. \tag{2.12}$$

In accordance with (2.6), we require that $\mathbf{u}_h$ satisfy

$$\sum_{\mathbf{x}\in\mathcal{G}_h} \mathbf{u}_h(\mathbf{x})h^2 = 0 \tag{2.13}$$

where $\mathcal{G}_h$ denotes the set of fluid grid points.

Following [36], we now write down an integral or sum representation of the solution to the continuous and discrete problems.

The periodic Stokes problem is well-studied (see, e.g., [12, 27, 28]). The solution of the continuous Stokes problem can be written as follows:

$$\mathbf{u}(\mathbf{x}) = \int_{-\pi}^{\pi} G(\mathbf{x} - \mathbf{X}(\theta))\mathbf{F}(\theta)d\theta \tag{2.14}$$

where G is a 2×2 matrix Green's function that corresponds to the Stokes problem on a two-dimensional periodic domain. The function G can be expressed as a lattice sum of the two-dimensional free-space Stokeslet (care must be taken so that the sum is convergent) or can be written as the following Fourier sum:

$$\begin{aligned} G(\mathbf{x}) &= \frac{1}{(2\pi)^2}\sum_{|\mathbf{k}|\neq 0}\exp(i\mathbf{k}\cdot\mathbf{x})\frac{1}{|\mathbf{k}|^2}\mathcal{P}_{\mathbf{k}}, \\ \mathbf{k} &= \begin{pmatrix} k \\ l \end{pmatrix}, \quad \mathcal{P}_{\mathbf{k}} = I - \frac{\mathbf{k}\mathbf{k}^{\mathsf{T}}}{|\mathbf{k}|^2} = \frac{1}{|\mathbf{k}|^2}\begin{pmatrix} l^2 & -kl \\ -kl & k^2 \end{pmatrix}. \end{aligned} \tag{2.15}$$

where I is the 2×2 identity matrix, and the components k and l of the vector $\mathbf{k}$ are integers. In the above and in what follows, we let $|\cdot|$ of a vector denote its Euclidean length. We shall often use the notation

$$G_{\mathbf{x}}(\mathbf{y}) \equiv G(\mathbf{x}-\mathbf{y}) = G(\mathbf{y}-\mathbf{x}).$$

Note that the constraint in (2.6) is already built into expression (2.14) since the Green's function G does not have a zero-frequency component.

We would now like to write down a similar expression for the discrete problem. Taking the discrete Fourier transform of equations (2.7)–(2.9) and taking a spectral discretization for the Laplacian and gradient/divergence operators, we find

$$\mathbf{u}_h(\mathbf{x}) = \sum_{\mathbf{y}\in\mathcal{G}_h} G_h(\mathbf{x}-\mathbf{y})\mathbf{f}_h(\mathbf{y})h^2. \tag{2.16}$$

We shall often make use of the notation $G_{h,\mathbf{x}}(\mathbf{y}) \equiv G_h(\mathbf{x}-\mathbf{y}) = G_h(\mathbf{y}-\mathbf{x})$. The function G_h is given by

$$G_h(\mathbf{x}) = \frac{1}{(2\pi)^2}\sum_{\mathbf{k}\in\mathcal{K}_h}\exp(i\mathbf{k}\cdot\mathbf{x})\frac{1}{|\mathbf{k}|^2}\mathcal{P}_{\mathbf{k}} \tag{2.17}$$

where the matrix $\mathcal{P}_{\mathbf{k}}$ is the same as in equation (2.15), and $\mathcal{K}_h$ is a square-shaped subset of the $\mathbf{k}$-plane,

$$\mathcal{K}_h = \{\mathbf{k} = (k,l)^{\mathsf{T}} \mid |\mathbf{k}| \neq 0,\ |k| < \pi/h,\ |l| < \pi/h\}. \tag{2.18}$$

We shall call G_h the discrete Green's function. The discrete Green's function G_h is nothing more than a band-limited version of the continuous Green's function G.

Now, consider the expression for $\mathbf{f}_h$:

$$\mathbf{f}_h(\mathbf{x}) = (S(\mathbf{X})\mathbf{F})(\mathbf{x}) \equiv \sum_{m=1}^{M}\mathbf{F}(\theta_m)\delta_h(\mathbf{x}-\mathbf{X}(\theta_m))\Delta\theta. \tag{2.19}$$

$S(\mathbf{X})$ can be seen as a linear operator that maps a function defined on the immersed boundary points to a function defined on the Eulerian grid. This operator $S(\mathbf{X})$, aside from $\mathbf{X}$, depends on h, $\Delta\theta$, and the choice of the discrete delta function δ_h.

We shall call this the *spreading operation*. Substituting this into (2.16), we find

$$\begin{aligned}\mathbf{u}_h(\mathbf{x}) &= \sum_{\mathbf{y}\in\mathcal{G}_h} G_{h,\mathbf{x}}(\mathbf{y})(S(\mathbf{X})\mathbf{F})(\mathbf{y})h^2 \\ &= \sum_{\mathbf{y}\in\mathcal{G}_h} G_{h,\mathbf{x}}(\mathbf{y})\left(\sum_{m=1}^{M}\mathbf{F}(\theta_m)\delta_h(\mathbf{y}-\mathbf{X}(\theta_m))\Delta\theta\right)h^2 \\ &= \sum_{m=1}^{M}\left(\sum_{\mathbf{y}\in\mathcal{G}_h} G_{h,\mathbf{x}}(\mathbf{y})\delta_h(\mathbf{y}-\mathbf{X}(\theta_m))h^2\right)\mathbf{F}(\theta_m)\Delta\theta.\end{aligned} \tag{2.20}$$

For any function q on the fluid grid, define S^* as follows:

$$(S^*(\mathbf{X})q)(\theta_m) = \sum_{\mathbf{x}\in\mathcal{G}_h} q(\mathbf{x})\delta_h(\mathbf{x}-\mathbf{X}(\theta_m))h^2. \tag{2.21}$$

This linear map $S^*(\mathbf{X})$ interpolates the function q defined on the fluid grid at the immersed boundary points $\mathbf{X}(\theta_m)$. We shall call $S^*(\mathbf{X})$ the *interpolation operator*. The operator $S^*(\mathbf{X})$ will also act on vectors and matrices, with the understanding that it acts componentwise. Note that $S^*(\mathbf{X})$ is the transpose of $S(\mathbf{X})$, as the notation suggests. We can now rewrite (2.20) as

$$\mathbf{u}_h(\mathbf{x}) = \sum_{m=1}^{M}(S^*(\mathbf{X})G_{h,\mathbf{x}})(\theta_m)\mathbf{F}(\theta_m)\Delta\theta. \tag{2.22}$$

Our task now is to understand under what conditions (2.22) converges to (2.14) and at what rate.

3 Discrete Delta Function

In this section, we discuss the discrete delta function and some properties of the linear operations that make use of the discrete delta function. A mathematically equivalent presentation of the facts to be shown below can be found in [2, 32, 36]. We also note that essentially equivalent results are obtained in [20], where these functions are used for image reconstruction. We include these results here nonetheless to present them in a form most conducive to our purposes.

We assume that δ_h has the form

$$\delta_h(\mathbf{x}) = \frac{1}{h^2}\phi\left(\frac{x}{h}\right)\phi\left(\frac{y}{h}\right) \tag{3.1}$$

for some function ϕ defined on the real line. Some of the commonly imposed conditions on ϕ are [25]:

- ϕ has compact support. We let the support of $\phi(r)$ satisfy $|r| \leq r_\phi$.
- ϕ satisfies moment conditions. When ϕ satisfies

$$\sum_{n\in\mathbb{Z}} \phi(n-r) = 1 \quad \text{if } p = 0, \tag{3.2}$$

$$\sum_{n\in\mathbb{Z}} (n-r)^p \phi(n-r) = 0 \quad \text{if } p > 0, \tag{3.3}$$

  for all $r \in \mathbb{R}$, ϕ is said to satisfy the p^{th}*-order moment condition*. If ϕ has compact support, the above sums contain only finitely many terms. If ϕ satisfies moment conditions up to order $p-1$, we shall say, following [36], that ϕ is of *moment order* p.
- ϕ satisfies the following *even-odd condition*:

$$\sum_{n \text{ even}} \phi(n-r) = \sum_{n \text{ odd}} \phi(n-r) \tag{3.4}$$

  for all $r \in \mathbb{R}$.
- ϕ is continuous.

The first condition is for computational efficiency. It ensures that each immersed boundary point only communicates with a finite number of grid points independent of the mesh spacing h. For any point $\mathbf{X}$ on the fluid domain, we define the following square-shaped region:

$$\mathcal{R}_\mathbf{X} = \left\{\mathbf{x} = (x, y)^\mathsf{T} \mid |x - X| \leq r_\phi h,\ |y - Y| \leq r_\phi h,\ \mathbf{X} = (X, Y)^\mathsf{T}\right\}. \tag{3.5}$$

$\mathcal{R}_\mathbf{X}$, which depends also on ϕ and h, is a subset of the fluid region with which an immersed boundary point located at $\mathbf{X}$ may communicate.

The moment conditions, as we shall see, determine the accuracy of the operations performed with the discrete delta function. The third condition was introduced to avoid the "checkerboard" type instability that manifests itself when using the central difference operator for the discretization of the ∇p and the $\nabla \cdot \mathbf{u}$ terms in the Stokes equations [25]. Here we are using a spectral scheme, so we do not expect this condition to play an important role. We shall, however, briefly come back to this condition in Section 8. The fourth condition will not be used in this paper except to ensure that ϕ remains bounded if ϕ has compact support. This condition may have important consequences especially in relation to the advection of the immersed boundary.

We henceforth assume that ϕ has compact support and is bounded, and is of moment order at least 1 (that is to say, ϕ satisfies the zeroth-order moment condition).

We now prove two lemmas about the interpolation operator S^*. The first one is trivial.

LEMMA 3.1 *Let $q(\mathbf{x})$ be a function defined on the Cartesian fluid domain. Suppose we are interpolating at a point $\mathbf{X_0} = \mathbf{X}(\theta_0)$. Then*

$$|(S^*(\mathbf{X})q)(\theta_0)| \leq C \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |q(\mathbf{x})| \tag{3.6}$$

for some constant C that depends only on ϕ.

PROOF: We have:

$$(S^*(\mathbf{X})q)(\theta_0) = \sum_{\mathbf{x}\in\mathcal{G}_h} \delta_h(\mathbf{x}-\mathbf{X_0})q(\mathbf{x})h^2. \tag{3.7}$$

Substitute (3.1) in the above:

$$(S^*(\mathbf{X})q)(\theta_0) = \sum_{\mathbf{x}\in\mathcal{G}_h\cap\mathcal{R}_{\mathbf{X}_0}} \phi\left(\frac{x-X_0}{h}\right)\phi\left(\frac{y-Y_0}{h}\right)q(\mathbf{x}) \tag{3.8}$$

where $\mathbf{X_0} = (X_0, Y_0)^\mathsf{T}$. By taking absolute values on both sides,

$$\begin{aligned} &|(S^*(\mathbf{X})q)(\theta_0)| \\ &\leq \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |q(\mathbf{x})| \sum_{\mathbf{x}\in\mathcal{G}_h\cap\mathcal{R}_{\mathbf{X}_0}} \left|\phi\left(\frac{x-X_0}{h}\right)\right| \left|\phi\left(\frac{y-Y_0}{h}\right)\right| \qquad (3.9) \\ &\leq \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |q(\mathbf{x})| \left(\max_{r\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\phi(n-r)|\right)^2. \end{aligned}$$

The sum in the last line is bounded from above since ϕ has compact support and is bounded. In fact, if ϕ is positive, the above sum is equal to 1 thanks to the zeroth-order moment condition. □

Our second lemma tells us that the accuracy of the interpolation operation is determined by the moment conditions.

LEMMA 3.2 *Let $q(\mathbf{x})$ be a C^n function defined on the Cartesian fluid domain, and ϕ be of moment order n. Suppose we interpolate q at a point $\mathbf{X_0} = (X_0, Y_0)^\mathsf{T} = \mathbf{X}(\theta_0)$,*

$$|((S^*(\mathbf{X})q)(\theta_0) - q(\mathbf{X_0})| \leq Ch^n \max_{\alpha+\beta=n} \big(\max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |\partial_x^\alpha \partial_y^\beta q(\mathbf{x})|\big) \tag{3.10}$$

where ∂_x^α denotes the α^{th} derivative in the x-direction and likewise for ∂_y^β. The above constant C depends only on ϕ and n.

PROOF: For any point $\mathbf{x} = (x, y)^\mathsf{T} \in \mathcal{R}_{\mathbf{X}_0}$, one can define the following function g:

$$g(t) = q(\mathbf{X_0} + t(\mathbf{x} - \mathbf{X_0})). \tag{3.11}$$

Note $g(0) = q(\mathbf{X}_0)$ and $g(1) = q(\mathbf{x})$. Applying the Taylor formula to this equation, we have

$$
\begin{aligned}
q(\mathbf{x}) &= q(\mathbf{X}_0) \\
&\quad + \sum_{k=1}^{n-1} \sum_{\alpha+\beta=k} \frac{1}{\alpha!\beta!} \partial_x^\alpha \partial_y^\beta q(\mathbf{X}_0)(x - X_0)^\alpha (y - Y_0)^\beta + r(\mathbf{x}), \\
r(\mathbf{x}) &= \sum_{\alpha+\beta=n} \frac{1}{\alpha!\beta!} \partial_x^\alpha \partial_y^\beta q(\mathbf{z}_\mathbf{x})(x - X_0)^\alpha (y - Y_0)^\beta, \\
\mathbf{z}_\mathbf{x} &= \mathbf{X}_0 + \theta(\mathbf{x} - \mathbf{X}_0)
\end{aligned}
\tag{3.12}
$$

where θ is some constant $0 < \theta < 1$ and depends on $\mathbf{x}$. Note that $\mathbf{z}_\mathbf{x} \in \mathcal{R}_{\mathbf{X}_0}$ since $\mathbf{z}_\mathbf{x}$ lies on the straight line between $\mathbf{x}$ and $\mathbf{X}_0$ and $\mathcal{R}_{\mathbf{X}_0}$ is convex.

First we have

$$
\begin{aligned}
(S^*(\mathbf{X})(1))(\theta_0) &= \sum_{x \in \mathcal{G}_h} \delta_h(\mathbf{x} - \mathbf{X}_0) h^2 \\
&= \sum_{x \in \mathcal{G}_h} \phi\left(\frac{x - X_0}{h}\right) \phi\left(\frac{y - Y_0}{h}\right) \\
&= \left(\sum_{m \in \mathbb{Z}} \phi\left(m - \frac{X_0}{h}\right)\right)\left(\sum_{m \in \mathbb{Z}} \phi\left(m - \frac{Y_0}{h}\right)\right) = 1
\end{aligned}
\tag{3.13}
$$

where we used the zeroth-order moment condition in the last line. Note for any $(\alpha, \beta) \neq (0, 0)$, $\alpha < n$, and $\beta < n$,

$$
\begin{aligned}
&(S^*(\mathbf{X})((x - X_0)^\alpha (y - Y_0)^\beta))(\theta_0) \\
&\quad = \sum_{x \in \mathcal{G}_h} \delta_h(\mathbf{x} - \mathbf{X}_0)(x - X_0)^\alpha (y - Y_0)^\beta h^2 \\
&\quad = \sum_{x \in \mathcal{G}_h} \phi\left(\frac{x - X_0}{h}\right) \phi\left(\frac{y - Y_0}{h}\right)(x - X_0)^\alpha (y - Y_0)^\beta \\
&\quad = \left(\sum_{m \in \mathbb{Z}} \phi\left(m - \frac{X_0}{h}\right)\left(m - \frac{X_0}{h}\right)^\alpha\right) \\
&\qquad \cdot \left(\sum_{m \in \mathbb{Z}} \phi\left(m - \frac{Y_0}{h}\right)\left(m - \frac{Y_0}{h}\right)^\beta\right) h^{\alpha+\beta} = 0
\end{aligned}
\tag{3.14}
$$

where we used the moment conditions in the last line. Applying S^* to (3.12) and using the above two calculations, we have

$$
(S^*(\mathbf{X})q)(\theta_0) = q(\mathbf{X}_0) + (S^*(\mathbf{X})r)(\theta_0). \tag{3.15}
$$

We compute $|(S^*(\mathbf{X})r)(\theta_0)|$. First,

$$
\begin{aligned}
&\left|\left(S^*(\mathbf{X})\left(\partial_x^\alpha \partial_y^\beta q(\mathbf{z_x})(x-X_0)^\alpha (y-Y_0)^\beta\right)\right)(\theta_0)\right| \\
&\quad\leq \left|\sum_{x\in\mathcal{G}_h} \partial_x^\alpha \partial_y^\beta q(\mathbf{z_x})\phi\left(\frac{x-X_0}{h}\right)\phi\left(\frac{y-Y_0}{h}\right)(x-X_0)^\alpha (y-Y_0)^\beta\right| \\
&\quad\leq \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |\partial_x^\alpha \partial_y^\beta q(\mathbf{z_x})| C_\alpha C_\beta h^n \leq \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |\partial_x^\alpha \partial_y^\beta q(\mathbf{x})| C_\alpha C_\beta h^n, \\
&C_{\alpha,\beta} = \max_{r\in\mathbb{R}}\left(\sum_{m\in\mathbb{Z}} |\phi(m-r)||m-r|^{\alpha,\beta}\right)
\end{aligned}
\tag{3.16}
$$

where we used $\mathbf{z_x}\in\mathcal{R}_{\mathbf{X}_0}$. Thus,

$$
\begin{aligned}
|(S^*(\mathbf{X})r)(\theta_0)| &\leq \sum_{\alpha+\beta=n} \frac{C_\alpha C_\beta}{\alpha!\,\beta!} \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |\partial_x^\alpha \partial_y^\beta q(\mathbf{x})| h^n \\
&= \left(\sum_{\alpha+\beta=n} \frac{C_\alpha C_\beta}{\alpha!\beta!}\right) h^n \max_{\alpha+\beta=n}\left(\max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0}} |\partial_x^\alpha \partial_y^\beta q(\mathbf{x})|\right).
\end{aligned}
\tag{3.17}
$$

This proves the assertion. □

We remark that if the function q in the above is only Lipschitz and ϕ is at least of moment order 1, we still have the following bound, which can be readily proved:

$$
|(S^*(\mathbf{X})q)(\theta_0) - q(\mathbf{X}_0)| \leq Ch \tag{3.18}
$$

where the constant C above depends only on the Lipschitz constant and the function ϕ. We shall use this fact later in Section 7.

4 Estimates for Green's Function

In this section we shall state some useful estimates for the continuous and discrete Green's function.

We first consider the continuous Green's function. All of the results in the following lemma are implicit in classical studies of the periodic Stokes problem [12].

LEMMA 4.1 *Suppose*

$$
\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad |x| \leq 2\pi - d,\ |y| \leq 2\pi - d,\ d > 0. \tag{4.1}
$$

Then, we have the following:

$$
|G_{ij}(\mathbf{x})| \leq C \log(|\mathbf{x}|^{-1}) + C, \tag{4.2}
$$

$$
|\partial_x^\alpha \partial_y^\beta G_{ij}(\mathbf{x})| \leq \frac{C}{|\mathbf{x}|^n} + C_d^n, \tag{4.3}
$$

where G_{ij} is the ij^{th} element of the Green's function matrix, and $\alpha+\beta=n$. The constant C depends only on n and C_d^n depends only on d and n.

PROOF: Define the set

$$D_d = \{\mathbf{x} \mid |x| \leq 2\pi - d,\ |y| \leq 2\pi - d\}. \tag{4.4}$$

Because our computational domain is a periodic square domain of side 2π, the Green's function G can be written as a lattice sum of the free space two-dimensional Stokeslets G^{free} (care must be taken so as to obtain a convergent sum).Thus, on D_d, G can be written as follows:

$$G(\mathbf{x}) = G^{\text{free}}(\mathbf{x}) + R(\mathbf{x}), \quad G^{\text{free}}(\mathbf{x}) = \frac{1}{4\pi}\left(I \log\left(\frac{1}{|\mathbf{x}|}\right) + \frac{\mathbf{x}\mathbf{x}^{\mathsf{T}}}{|\mathbf{x}|^2}\right) \tag{4.5}$$

where $R(\mathbf{x})$ is a smooth function and I is the identity matrix. Taking any n^{th} partial derivative of the above,

$$\partial_x^\alpha \partial_y^\beta G_{ij}(\mathbf{x}) = \partial_x^\alpha \partial_y^\beta G_{ij}^{\text{free}}(\mathbf{x}) + \partial_x^\alpha \partial_y^\beta R_{ij}(\mathbf{x}). \tag{4.6}$$

Since R is smooth up to the boundary of D_d, its n^{th} partial derivatives are bounded by a universal constant C_d^n. The derivatives of G^{free} can be computed explicitly, and we see that each element is bounded by $C/|x|^n$ for a constant C that only depends on n. □

We turn to the discrete Green's function G_h. We start with the following trivial bound:

LEMMA 4.2

$$|G_{h,ij}(\mathbf{x})| \leq C \log(h^{-1}) \tag{4.7}$$

for a universal constant C, where $G_{h,ij}$ denotes the ij^{th} element of the discrete Green's function matrix.

PROOF: Take (2.17):

$$|G_{h,ij}(\mathbf{x})| \leq \left|\sum_{\mathbf{k}\in\mathcal{K}_h} \exp(i\mathbf{k}\cdot\mathbf{x})\frac{1}{|\mathbf{k}|^2}\mathcal{P}_{\mathbf{k},ij}\right| \leq \sum_{\mathbf{k}\in\mathcal{K}_h} \frac{1}{|\mathbf{k}|^2}|\mathcal{P}_{\mathbf{k},ij}|. \tag{4.8}$$

Noting that $\mathcal{P}_{\mathbf{k},ij}$ is bounded in absolute value by 1, we see that

$$|G_{h,ij}(\mathbf{x})| \leq \sum_{\mathbf{k}\in\mathcal{K}_h} \frac{1}{|\mathbf{k}|^2} \leq C\int_1^{\sqrt{2}\pi/h} \frac{1}{r^2}\, r\, dr \leq C \log(h^{-1}) \tag{4.9}$$

where we used $\mathbf{k} = (k, l)^{\mathsf{T}}$, $|k| < \pi/h$, and $|l| < \pi/h$ for $\mathbf{k} \in \mathcal{K}_h$. The last constant C is a universal constant (not all constants in the above may be equal, as will be the case elsewhere in the paper). □

We now prove estimates that tell us how well the discrete Green's function approximates the continuous Green's function.

Consider expressions (2.15) and (2.17) for the continuous and discrete Green's function. A first attempt to estimate the difference between the two might be the following:

$$(4.10)\qquad \begin{aligned} |G_{ij}(\mathbf{x}) - G_{h,ij}(\mathbf{x})| &= \left| \sum_{\mathbf{k}\neq 0, \mathbf{k}\notin \mathcal{K}_h} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{1}{|\mathbf{k}|^2} \mathcal{P}_{\mathbf{k},ij} \right| \\ &\leq \sum_{\mathbf{k}\neq 0, \mathbf{k}\notin \mathcal{K}_h} \frac{1}{|\mathbf{k}|^2} |\mathcal{P}_{\mathbf{k},ij}| = \sum_{|k|,|l|\geq \pi/h} \frac{1}{k^2+l^2}. \end{aligned}$$

The difficulty is that the last sum is not convergent. The above calculation indicates that the Fourier sum representation (2.15) of G is nowhere absolutely convergent. Equation (2.15) is an expression valid only in the L^2 sense. This difficulty comes from the fact that the free space two-dimensional Stokeslet (and hence the Green's function G) possesses a logarithmic singularity at the origin. We have thus a rather delicate summation problem that we must understand.

We start by estimating the magnitude of the following expression:

$$(4.11)\qquad G_K^L = \frac{1}{(2\pi)^2} \sum_{\mathbf{k}\in \mathcal{B}_K^L} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{1}{|\mathbf{k}|^2} \mathcal{P}_{\mathbf{k}}$$

where $\mathcal{B}_K^L$ is a square-toric shaped subset of the **k**-plane

$$(4.12)\qquad \mathcal{B}_K^L = \{\mathbf{k} \mid K \leq |k| \leq L,\ K \leq |l| \leq L\}.$$

LEMMA 4.3 *Suppose*

$$(4.13)\qquad \mathbf{x} = (x,y)^{\mathsf{T}} \neq \mathbf{0}, \quad |x| < 2\pi,\ |y| < 2\pi.$$

The ij^{th} element of G_K^L is bounded by

$$(4.14)\qquad \left|G_{K,ij}^L(\mathbf{x})\right| \leq \frac{C}{Kw(\mathbf{x})}, \quad w(\mathbf{x}) = \max\left(\sin\left(\frac{|x|}{2}\right), \sin\left(\frac{|y|}{2}\right)\right)$$

where the constant C is a universal constant and $\max(a,b)$ *denotes the greater value of a and b.*

PROOF: Assume that $|x| \neq 0$ and thus $0 < |x| < 2\pi$. Consider the sum

$$(4.15)\qquad \begin{gathered} S_{P,Q}^l = \sum_{k=P}^{Q} \alpha^k \beta^l \gamma_{kl}, \\ \alpha = \exp(ix),\ \beta = \exp(iy),\ \gamma_{kl} = \frac{1}{|\mathbf{k}|^2} \mathcal{P}_{\mathbf{k},ij}, \end{gathered}$$

for some P and Q. Note that l is fixed in the above sum. Multiply by $(1-\alpha)$ and rearrange the above sum:

$$\begin{aligned}(1-\alpha)S^l_{P,Q} &= \beta^l \sum_{k=P}^{Q} (\alpha^k - \alpha^{k+1})\gamma_{kl} \\ &= \beta^l \left(\alpha^P \gamma_{Pl} - \alpha^{Q+1}\gamma_{Ql} + \sum_{k=P+1}^{Q} \alpha^k (\gamma_{kl} - \gamma_{k-1,l})\right).\end{aligned} \tag{4.16}$$

The reader will recognize the above as a summation-by-parts procedure. By assumption, $0 < |x| < 2\pi$, and thus $\alpha \neq 1$. We can therefore divide both sides of (4.16) by $1-\alpha$:

$$S^l_{P,Q} = \beta^l \frac{\alpha^P \gamma_{Pl} - \alpha^{Q+1}\gamma_{Ql}}{1-\alpha} + \beta^l \sum_{k=P+1}^{Q} \frac{\alpha^k}{1-\alpha}(\gamma_{kl} - \gamma_{k-1,l}). \tag{4.17}$$

We take the absolute value of both sides:

$$|S^l_{P,Q}| \leq \frac{1}{|1-\alpha|}\left(|\gamma_{Pl}| + |\gamma_{Ql}| + \sum_{k=P+1}^{Q} |\gamma_{kl} - \gamma_{k-1,l}|\right) \tag{4.18}$$

where we used $|\beta| = 1$ and $|\alpha| = 1$. We have

$$|\gamma_{Pl}| \leq \frac{1}{|\mathbf{k}|^2_{k=P}} \leq \frac{1}{|P|^2}, \quad |\gamma_{Ql}| \leq \frac{1}{|\mathbf{k}|^2_{k=Q}} \leq \frac{1}{|Q|^2}. \tag{4.19}$$

We can also check that

$$|\gamma_{kl} - \gamma_{k-1,l}| \leq \frac{C}{|\mathbf{k}|^3} \tag{4.20}$$

where C is a universal constant. Thus,

$$|S^l_{P,Q}| \leq \frac{C}{|1-\alpha|}\left(\frac{1}{|P|^2} + \frac{1}{|Q|^2} + \sum_{k=P}^{Q} \frac{1}{|\mathbf{k}|^3}\right). \tag{4.21}$$

We now use this to estimate the magnitude of $G^L_{K,ij}$. $G^L_{K,ij}$ can be written as

$$G^L_{K,ij} = \sum_{l=-L}^{-K} S^l_{-L,L} + \sum_{l=-K+1}^{K-1} (S^l_{-L,-K} + S^l_{K,L}) + \sum_{l=K}^{L} S^l_{-L,L}. \tag{4.22}$$

Thus,

$$
\begin{aligned}
|G^L_{K,ij}| &\le \sum_{l=-L}^{-K} |S^l_{-L,L}| \\
&\quad + \sum_{l=-K+1}^{K-1} (|S^l_{-L,-K}| + |S^l_{K,L}|) + \sum_{l=K}^{L} |S^l_{-L,L}| \\
&\le \frac{C}{|1-\alpha|}\Bigg(\sum_{l=-L}^{L} \left(\frac{1}{L^2} + \frac{1}{L^2}\right) \\
&\qquad + \sum_{l=-K+1}^{K-1} \left(\frac{1}{K^2} + \frac{1}{K^2}\right) + \sum_{k\in\mathcal{B}^L_K} \frac{1}{|\mathbf{k}|^3}\Bigg) \\
&\le \frac{C}{|1-\alpha|}\left(\frac{8}{K} + \sum_{k\in\mathcal{B}^L_K} \frac{1}{|\mathbf{k}|^3}\right)
\end{aligned} \tag{4.23}
$$

where we used (4.21) to estimate the above sum and used $L > K$ to get the last line. For the last sum, we have

$$
\sum_{k\in\mathcal{B}^L_K} \frac{1}{|\mathbf{k}|^3} \le C \int_K^{\sqrt{2}\,L} \frac{1}{r^3}\, r\, dr \le \frac{C}{K} \tag{4.24}
$$

for some constant C. Putting this back into (4.23) and noting that $|1-\alpha| = 2\sin(|x|/2)$, we have

$$
|G^L_{K,ij}| \le \frac{C}{K\sin(|x|/2)}. \tag{4.25}
$$

If $y \neq 0$, we can likewise prove the bound

$$
|G^L_{K,ij}| \le \frac{C}{K\sin(|y|/2)}. \tag{4.26}
$$

Since $\mathbf{x} \neq 0$, at least one of $x \neq 0$ or $y \neq 0$ is true, and thus the desired conclusion follows. □

We can go on to prove a similar but stronger bound on G^L_K if $x \neq 0$ and $y \neq 0$. The idea of the proof is that if $x \neq 0$ *and* $y \neq 0$, one can perform the above summation-by-parts argument in each coordinate to obtain a better bound. We shall relegate its proof to Appendix A.

LEMMA 4.4 *Suppose*

$$
\mathbf{x} = (x, y)^{\mathsf{T}}, \quad 0 < |x| < 2\pi,\ 0 < |y| < 2\pi. \tag{4.27}
$$

The ij^{th} element of G_K^L is bounded by

$$|G^L_{K,ij}(\mathbf{x})| \le \frac{C}{K^2 \sin(|x|/2)\sin(|y|/2)} \tag{4.28}$$

where the constant C is a universal constant.

With the use of the two lemmas above, we can prove the following estimate on the difference between the discrete and continuous Green's functions:

PROPOSITION 4.5 *Suppose*

$$\mathbf{x} = (x,y)^{\mathsf{T}} \neq \mathbf{0}, \quad |x| < 2\pi,\ |y| < 2\pi. \tag{4.29}$$

Then $G_{h,ij}(\mathbf{x})$ converges to $G_{ij}(\mathbf{x})$. Furthermore, the difference between G and G_h satisfies the bound

$$|G_{ij}(\mathbf{x}) - G_{h,ij}(\mathbf{x})| \le \frac{Ch}{w(\mathbf{x})}, \quad w(\mathbf{x}) = \max\left(\sin\left(\frac{|x|}{2}\right), \sin\left(\frac{|y|}{2}\right)\right). \tag{4.30}$$

If $x \neq 0$ and $y \neq 0$, the difference satisfies a stronger bound:

$$|G_{ij}(\mathbf{x}) - G_{h,ij}(\mathbf{x})| \le \frac{Ch^2}{\sin(|x|/2)\sin(|y|/2)} \tag{4.31}$$

where the constants C in the above do not depend on $\mathbf{x}$ *or h.*

PROOF: Define the square-shaped subset $\mathcal{A}_K$ in the $\mathbf{k}$-plane:

$$\mathcal{A}_K = \mathcal{B}_1^K = \left\{\mathbf{k} = (k,l)^{\mathsf{T}} \mid 0 < |k| \le K,\ 0 < |l| \le K\right\}. \tag{4.32}$$

Define G^K as

$$G^K(\mathbf{x}) \equiv \sum_{\mathbf{k}\in\mathcal{A}_K} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{1}{|\mathbf{k}|^2} \mathcal{P}_{kl}. \tag{4.33}$$

We now prove that $G^K(\mathbf{x})$ converges to $G(\mathbf{x})$. Let $B = \{\mathbf{x} \mid \mathbf{x} \neq 0,\ |x| < 2\pi,\ |y| < 2\pi\}$. Take any point $\mathbf{x_0}$ in B and a closed disc D of radius $\epsilon > 0$ centered at $\mathbf{x_0}$ so that this disc is contained in B. For any point $\mathbf{x}$ in D,

$$\left|G^L_{ij}(\mathbf{x}) - G^K_{ij}(\mathbf{x})\right| = \left|G^L_{K+1,ij}(\mathbf{x})\right| \le \frac{C}{(K+1)w(\mathbf{x})}, \tag{4.34}$$

where we have assumed $L > K$ and used Lemma 4.3. The function w is given in the statement of this lemma. By taking the maximum on both sides with respect to $\mathbf{x}$, we have

$$\max_{\mathbf{x}\in D}\left|G^L_{ij}(\mathbf{x}) - G^K_{ij}(\mathbf{x})\right| \le \frac{C_D}{K+1} \tag{4.35}$$

where C_D is a constant that depends on the disc D. This shows that G^K_{ij} is a Cauchy sequence in the maximum norm on this disc and thus converges to some continuous function $\tilde{G}_{ij}$ on D. On the other hand, it is clear from expression (4.33) and (2.15) that $G^K_{ij} \to G_{ij}$ in the L^2 norm on the periodic unit cell (a coordinate square of side 2π). If we let χ_D denote the characteristic function of the disc D, it follows that $\chi_D G^K_{ij} \to \chi_D G_{ij}$ in the L^2 norm. This shows that $\tilde{G}_{ij}$ should be equal to the continuous version of the L^2 function G_{ij} on D. This proves that $G^K_{ij}(\mathbf{x}) \to G_{ij}(\mathbf{x})$ on all points of D. Since the center of the disc D was arbitrary, $G^K_{ij}(\mathbf{x}) \to G_{ij}(\mathbf{x})$ for all $\mathbf{x} \in B$.

Now we can take $L \to \infty$ in (4.34) to find

$$\left|G_{ij}(\mathbf{x}) - G^K_{ij}(\mathbf{x})\right| \le \frac{C}{(K+1)w(\mathbf{x})}. \tag{4.36}$$

Noting that $G_h = G^K$ for $K < \pi/h \le K+1$, we find

$$|G_{ij}(\mathbf{x}) - G_{h,ij}(\mathbf{x})| \le \frac{Ch}{w(\mathbf{x})}. \tag{4.37}$$

If $x \neq 0$ and $y \neq 0$, we have from Lemma 4.4

$$\left|G^L_{ij}(\mathbf{x}) - G^K_{ij}(\mathbf{x})\right| = \left|G^L_{K+1,ij}(\mathbf{x})\right| \le \frac{C}{(K+1)^2 \sin(|x|/2)\sin(|y|/2)}. \tag{4.38}$$

Taking $L \to \infty$ and noting $G_h = G^K$ and $K < \pi/h \le K+1$, we have

$$|G_{ij}(\mathbf{x}) - G_{h,ij}(\mathbf{x})| \le \frac{Ch^2}{\sin(|x|/2)\sin(|y|/2)}. \tag{4.39}$$

□

It is interesting to note that, although we use a spectral scheme to discretize the fluid equations, the discrete Green's function converges to the continuous Green's function only at an order 1 or order 2 rate. This is, after all, hardly surprising given that the Green's function represents the solution to the Stokes problem with a delta function as the external force field. We can only expect spectral convergence if the external force field is a smooth function.

Where does the convergence rate of 2 come from? By looking through the above argument, we realize that this rate comes from the fact that the force field **f** "gains" two derivatives when the Stokes problem is solved to find the velocity field **u**. The order 1 rate seen where $x = 0$ or $y = 0$ in Proposition 4.5 may be understood as coming from grid effects.

5 Local Error Estimates

We are now ready to prove error estimates about the immersed boundary solution. In this section, we concern ourselves with pointwise error estimates valid at points away from the immersed boundary. We assume that the immersed boundary curve $\Gamma = \{\mathbf{x} \in \mathbb{U} \mid \mathbf{x} = \mathbf{X}(\theta)\}$ is at least a distance $d_{\mathbb{U}} > 0$ away from the boundary of $\mathbb{U}$.

In the following theorem, we estimate the difference between the exact and approximate solutions by dividing it into three parts, a quadrature error term, an interpolation error term, and an error term due to the discretization of the fluid domain (the difference between the continuous and discrete Green's functions). This idea is based in part on a discussion given in [36].

THEOREM 5.1 *Let ϕ be of moment order p, and $\mathbf{F}(\theta)$ and $\mathbf{X}(\theta)$ be C^n as functions of θ. For $\mathbf{x} \notin \Gamma$, the following error bound holds for sufficiently small h and $\Delta\theta$:*

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \leq C(h + h^{\min(p,n)} + (\Delta\theta)^n). \tag{5.1}$$

If no points on Γ share the same x- or y-coordinate as the point $\mathbf{x}$, $\mathbf{u}_h$ satisfies the stronger error estimate

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \leq C(h^2 + h^{\min(p,n)} + (\Delta\theta)^n) \tag{5.2}$$

for h and $\Delta\theta$ small enough. The constants C in the above do not depend on h or $\Delta\theta$ but may depend on $\mathbf{x}$.

PROOF: We first prove (5.1). Suppose

$$\operatorname{dist}(\mathbf{x}, \Gamma) \equiv \min_\theta |\mathbf{x} - \mathbf{X}(\theta)| \geq d_\Gamma > 0. \tag{5.3}$$

Consider the difference between (2.14) and (2.22):

$$\begin{aligned} \mathbf{u} - \mathbf{u}_h &= \mathbf{I}^\int + \mathbf{I}^\delta + \mathbf{I}^G, \\ \mathbf{I}^\int &= \int_{-\pi}^{\pi} G_\mathbf{x}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - \sum_{m=1}^{M} G_\mathbf{x}(\mathbf{X}(\theta_m))\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{I}^\delta &= \sum_{m=1}^{M} (G_\mathbf{x}(\mathbf{X}(\theta_m)) - (S^*(\mathbf{X})G_\mathbf{x})(\theta_m))\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{I}^G &= \sum_{m=1}^{M} (S^*(\mathbf{X})(G_\mathbf{x} - G_{h,\mathbf{x}}))(\theta_m)\mathbf{F}(\theta_m)\Delta\theta. \end{aligned} \tag{5.4}$$

The term $\mathbf{I}^\int$ is a quadrature error. The integrand is a periodic function, and the quadrature method is the trapezoidal rule. Recalling that the trapezoidal rule is spectrally accurate for periodic functions, we have

$$|\mathbf{I}^\int| \leq C^\int (\Delta\theta)^n \tag{5.5}$$

where the constant $C^\int$ depends on the derivatives of $G_\mathbf{x}(\mathbf{X}(\theta))\mathbf{F}(\theta)$ with respect to θ. We know that $C^\int$ is bounded by Lemma 4.1 and the assumption that $\mathbf{F}(\theta)$ and $\mathbf{X}(\theta)$ are C^n. From equation (4.3) of Lemma 4.1, we see that $C^\int$ depends on $\mathbf{x}$ only through d_Γ.

We now estimate $\mathbf{I}^\delta$. Let h be small enough so that $\sqrt{2}\, r_\phi h < d_\Gamma/2$. Let $\mathbf{I}^\delta = (I_1^\delta, I_2^\delta)^\mathsf{T}$, $\mathbf{F} = (F_1, F_2)^\mathsf{T}$, and $q = \min(p, n)$.

$$
\begin{aligned}
|I_i^\delta| &\le \sum_{m=1}^{M}\sum_{j=1}^{2} |(G_{\mathbf{x},ij}(\mathbf{X}(\theta_m)) - (S^*(\mathbf{X})G_{\mathbf{x},ij})(\theta_m))F_j(\theta_m)|\Delta\theta \\
&\le C\sum_{m=1}^{M}\sum_{j=1}^{2} \max_{\alpha+\beta=q}\big(\max_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta_m)}} |\partial_x^\alpha \partial_y^\beta G_{\mathbf{x}}(\mathbf{y})|\big) h^q |F_j(\theta_m)|\Delta\theta \qquad (5.6)\\
&\le C\sum_{m=1}^{M} \max_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta_m)}} \left(\frac{C}{|\mathbf{x}-\mathbf{y}|^q} + C_d^q\right) h^q |\mathbf{F}(\theta_m)|\Delta\theta
\end{aligned}
$$

where we used Lemma 3.2 in the second inequality and Lemma 4.1 in the third. Note that

$$
|\mathbf{x}-\mathbf{y}| \ge |\mathbf{x}-\mathbf{X}(\theta)| - |\mathbf{y}-\mathbf{X}(\theta)| \ge d_\Gamma - \sqrt{2}\, r_\phi h \ge \frac{d_\Gamma}{2} \qquad (5.7)
$$

where we used (5.3) and $y \in \mathcal{R}_{\mathbf{X}(\theta)}$. Thus,

$$
\begin{aligned}
|I_i^\delta| &\le C\left(\frac{C}{(d_\Gamma/2)^q} + C_d^q\right) h^q \sum_{m=1}^{M} |\mathbf{F}(\theta_m)|\Delta\theta \qquad (5.8)\\
&\le C^\delta(\phi, d_\Gamma, \max_\theta |\mathbf{F}|, q) h^q.
\end{aligned}
$$

We now estimate $\mathbf{I}^G$. We take h small enough so that $\sqrt{2}\, r_\phi h < d_\Gamma/2$ and $r_\phi h < d_\mathbb{U}/2$. Let $\mathbf{I}^G = (I_1^G, I_2^G)^\mathsf{T}$,

$$
\begin{aligned}
|I_i^G| &\le \sum_{m=1}^{M}\sum_{j=1}^{2} |S^*(\mathbf{X})(G_{\mathbf{x},ij} - G_{h,\mathbf{x},ij})(\theta_m) F_j(\theta_m)|\Delta\theta \\
&\le \sum_{m=1}^{M}\sum_{j=1}^{2} \max_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta_m)}} C|(G_{\mathbf{x},ij}(\mathbf{y}) - G_{h,\mathbf{x},ij}(\mathbf{y}))||F_j(\theta_m)|\Delta\theta \qquad (5.9)\\
&\le \sum_{m=1}^{M} \max_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta_m)}} \frac{Ch}{w(\mathbf{x}-\mathbf{y})} |\mathbf{F}(\theta_m)|\Delta\theta
\end{aligned}
$$

where we have used Lemma 3.1 in the second inequality and Lemma 4.5 in the third. Let $\mathbf{x} = (x_0, y_0)^\mathsf{T}$, $\mathbf{y} = (x_1, y_1)^\mathsf{T}$, $\mathbf{z} = \mathbf{x} - \mathbf{y} = (x_2, y_2)^\mathsf{T}$, and $\mathbf{X}(\theta) = (X(\theta), Y(\theta))^\mathsf{T}$. We first note that

$$
\begin{aligned}
|x_2| = |x_0 - x_1| &\le |x_0 - X(\theta)| + |x_1 - X(\theta)| \qquad (5.10)\\
&\le 2\pi - d_\mathbb{U} + r_\phi h \le 2\pi - \frac{d_\mathbb{U}}{2}
\end{aligned}
$$

where we have used $\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta)}$ and so $|x_1 - X(\theta)| \leq r_\phi h$. An identical inequality holds for $|y_2| = |y_0 - y_1|$. We also have

$$|\mathbf{z}| = |\mathbf{x} - \mathbf{y}| \geq \frac{d_\Gamma}{2} \tag{5.11}$$

where we can argue as in (5.7). Thus the argument $\mathbf{z}$ of the function $w(\mathbf{z}) = w(\mathbf{x} - \mathbf{y})$ in (5.9) only ranges over the following domain $D_\mathbf{z}$:

$$D_\mathbf{z} = \left\{ \mathbf{z} = (x_2, y_2)^\mathsf{T} \;\middle|\; |\mathbf{z}| \geq \frac{d_\Gamma}{2},\ x_2 \leq 2\pi - \frac{d_\mathbb{U}}{2},\ y_2 \leq 2\pi - \frac{d_\mathbb{U}}{2} \right\}. \tag{5.12}$$

Since $w(\mathbf{z})$ is never 0 and is a continuous function that is always positive on this compact set $D_\mathbf{z}$, $w(\mathbf{z}) > C(d_\mathbb{U}, d_\Gamma) > 0$ where C is a constant that depends on $d_\mathbb{U}$ and d_Γ. Substituting this back into (5.9), we have

$$|I_i^G| \leq Ch \sum_{m=1}^{M} |\mathbf{F}(\theta_m)| \Delta\theta \leq C^G(d_\mathbb{U}, d_\Gamma, \max_\theta |\mathbf{F}|) h. \tag{5.13}$$

Collecting (5.5), (5.8), and (5.13), we obtain (5.1).

We now turn to (5.2). Let $\mathbf{x} = (x_0, y_0)$, and suppose the lines $x = x_0$ and $y = y_0$ never touch or intersect Γ. Define

$$d_x = \min_\theta |x_0 - X(\theta)|, \quad d_y = \min_\theta |y_0 - Y(\theta)|, \quad d_\mathbf{x} = \min(d_x, d_y). \tag{5.14}$$

A similar proof to that of inequality (5.13) yields

$$|I_i^G| \leq C(d_\mathbb{U}, d_\mathbf{x}, \max_\theta |\mathbf{F}|) h^2 \tag{5.15}$$

where instead of equation (4.30) in Proposition 4.5, equation (4.31) is used. Collecting (5.5), (5.8), and (5.15), we obtain the bound (5.2). □

We make the following observation. Fix a positive number d_Γ and consider all $\mathbf{x}$ such that $\mathrm{dist}(\mathbf{x}, \Gamma) \geq d_\Gamma$. Then, for any such $\mathbf{x}$ the constant in (5.1) can be taken uniformly in $\mathbf{x}$, since the constants $C^\int$, C^δ, and C^G in (5.5), (5.8), and (5.13) only depend on d_Γ and not directly on $\mathbf{x}$. In particular, if we take $p = 1$ and $n = 1$ we have

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \leq C(h + \Delta\theta) \quad \text{for } \mathbf{x} \text{ such that } \mathrm{dist}(\mathbf{x}, \Gamma) \geq d_\Gamma \tag{5.16}$$

where the constant C can be taken uniformly in $\mathbf{x}$.

We now prove the following generalization of (5.16), which we shall need later in the proof of Theorem 6.3. Consider a connected subset Γ_a of the closed curve Γ; Γ_a would thus be a filament with endpoints. Define

$$\begin{aligned} \mathbf{v} &= \int_{\mathbf{X}(\theta) \in \Gamma_a} G_\mathbf{x}(\mathbf{X}(\theta)) \mathbf{F}(\theta) d\theta, \\ \mathbf{v}_h &= \sum_{\mathbf{X}(\theta_m) \in \Gamma_a} (S^*(\mathbf{X}) G_{h,\mathbf{x}})(\theta_m) \mathbf{F}(\theta_m) \Delta\theta. \end{aligned} \tag{5.17}$$

The only difference between $\mathbf{v}$, $\mathbf{v}_h$ and $\mathbf{u}$, $\mathbf{u}_h$ is that the integration or summation is over Γ_a and not Γ. We have the following corollary for the difference between $\mathbf{v}$ and $\mathbf{v}_h$:

COROLLARY 5.2 *Suppose* $\mathbf{F}$ *and* $\mathbf{X}$ *are* C^1 *functions in* θ*, and* ϕ *is of moment order at least* 1*. Fix a positive number* d_Γ *and consider* $\mathbf{x}$ *such that* $\operatorname{dist}(\mathbf{x}, \Gamma_a) \geq d_\Gamma > 0$*. The following holds*:

$$|\mathbf{v}(\mathbf{x}) - \mathbf{v}_h(\mathbf{x})| \leq C(h + \Delta\theta) \tag{5.18}$$

where the constant C *may be taken to be the same as the constant in* (5.16)*. Thus, the constant* C *does not depend on* $\mathbf{x}$*,* $\operatorname{dist}(\mathbf{x}, \Gamma_a) \geq d_\Gamma$*, or the choice of connected subset* Γ_a*.*

PROOF: We estimate the difference $\mathbf{v}-\mathbf{v}_h$ by dividing it into three parts, $\mathbf{I}_v^{\int}$, $\mathbf{I}_v^{\delta}$, and $\mathbf{I}_v^{G}$, which correspond respectively to $\mathbf{I}^{\int}$, $\mathbf{I}^{\delta}$, and $\mathbf{I}^{G}$ in equation (5.4). Since Γ_a is a subset of Γ, it is clear that inequalities (5.8) and (5.13) hold for $\mathbf{I}_v^{\delta}$ and $\mathbf{I}_v^{G}$ where we take $q = 1$, with the same constants C^δ and C^G.

The fact that Γ is a closed filament allowed us to use the properties of the trapezoidal rule for quadrature of periodic functions in the proof of (5.5). Since Γ_a is not a closed filament, (5.5) does not hold for $\mathbf{I}_v^{\int}$ when $n > 1$, but does still hold for $n = 1$ with the same constant $C^{\int}$. Combining the estimates for $\mathbf{I}_v^{\int}$, $\mathbf{I}_v^{\delta}$, and $\mathbf{I}_v^{G}$, we obtain the desired estimate. □

We proved in Theorem 5.1 that at points away from the immersed boundary, the generally applicable error estimate is (5.1), whereas if the point $\mathbf{x_0} = (x_0, y_0)^{\mathsf{T}}$ does not share the x- or the y-coordinate with Γ, (5.2) holds. For points that do not share an x- or y-coordinate, convergence is considerably faster, according to Theorem 5.1. However, the following is true:

Suppose $x = x_0$ and $y = y_0$ is never tangent to Γ. We shall call such points $\mathbf{x_0} = (x_0, y_0)^{\mathsf{T}}$ *nontangent points*. For nontangent points, we can prove the following bound, whose proof is given in Appendix B.

THEOREM 5.3 *Suppose* $\mathbf{X}$ *and* $\mathbf{F}$ *satisfy the conditions of Theorem* 5.1 *and, in addition,* $|\partial\mathbf{X}/\partial\theta| \neq 0$*. Nontangent points are generic*: *almost all points* $\mathbf{x} \in \mathbb{U}\backslash\Gamma$*, in the sense of Lebesgue measure, are nontangent points. For nontangent* $\mathbf{x}$*, the following error bound holds for small enough* h *and* $\Delta\theta$:

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \leq C\big(h(\Delta\theta + h\log(h^{-1})) + h^{\min(p,n)} + (\Delta\theta)^n\big) \tag{5.19}$$

where the constant C *may depend on* $\mathbf{x}$ *but not on* h *or* $\Delta\theta$*.*

Since (5.19) holds almost everywhere in $\mathbb{U}\backslash\Gamma$, we take this as the representative error bound that holds for points in the computational domain. Suppose we let

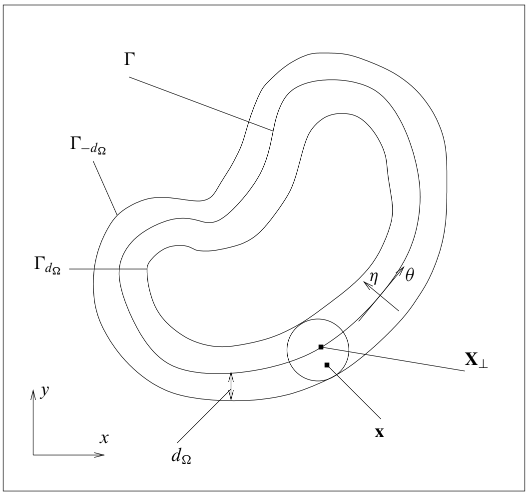

FIGURE 6.1. Schematic diagram of the geometric constructs used in the proof of Theorem 6.3. The region Ω is the region between the innermost and outermost curves, which are both a distance d_Ω away from Γ. When the point $\mathbf{x}$ is within the region Ω, we consider the closest point on Γ, which we denote by $\mathbf{X}_\perp$, and a circle of radius d_Ω around $\mathbf{X}_\perp$.

$\Delta\theta \sim h^\alpha$ as $h \to 0$ and $\mathbf{X}$ and $\mathbf{F}$ are smooth so that n can be taken arbitrarily large. Then the error estimate (5.19) reduces to

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \le C\big(h(h^\alpha + h\log(h^{-1})) + h^p\big). \tag{5.20}$$

The best convergence rate that can be obtained with the above estimate is for $\alpha \ge 1$ and $p \ge 2$, in which case the asymptotic convergence rate is $h^2\log(h^{-1})$. We shall see in Section 8 that this estimate seems to be optimal (modulo logarithmic factors) except for certain types of the discrete delta functions. We shall come back to this point at the end of Section 8.

6 Global Error Estimate

We now turn to a global error estimate valid everywhere including points at or close to the immersed boundary. Such bounds are important not least because the velocity field of the immersed boundary solution is interpolated at the immersed boundary points. In order to obtain this bound, we require some geometric constructs. We assume in the rest of this section that $\mathbf{X}$ is at least C^2 as a function of θ and that $|\partial\mathbf{X}/\partial\theta| \ne 0$.

We introduce a local coordinate system in the vicinity Ω of the curve Γ (Figure 6.1):

$$\Omega = \{\mathbf{x} \mid \operatorname{dist}(\mathbf{x},\Gamma) \equiv \min_\theta |\mathbf{x} - \mathbf{X}(\theta)| < d_\Omega\}. \tag{6.1}$$

We must specify d_Ω as well as the local coordinate on Ω.

Since Γ is a simple closed curve, it separates $\mathbb{U}$ into two regions. Given the tangent vector $\partial\mathbf{X}/\partial\theta \neq \mathbf{0}$ on Γ, we let $\mathbb{U}_1$ be the region to the left of this tangent vector and $\mathbb{U}_2$ to the right. Define the signed-distance function $\tilde{\eta}$:

$$\tilde{\eta}(\mathbf{x}) = \begin{cases} \operatorname{dist}(\mathbf{x}, \Gamma) & \text{if } \mathbf{x} \in \mathbb{U}_1, \\ 0 & \text{if } \mathbf{x} \in \Gamma, \\ -\operatorname{dist}(\mathbf{x}, \Gamma) & \text{if } \mathbf{x} \in \mathbb{U}_2. \end{cases} \tag{6.2}$$

Denote by Γ_{η_0} the set of points (*level set*) for which $\tilde{\eta} = \eta_0$. We note $\Gamma_0 = \Gamma$. There is a real number $d_\eta > 0$ such that if $|\eta_0| < d_\eta$, Γ_{η_0} is a simple closed curve. We require that $d_\Omega < d_\eta$.

Now define the local coordinate system as a map $\mathbf{Z}$ from Ξ to Ω:

$$\begin{aligned} \mathbf{Z}(\boldsymbol{\xi}) &= (Z(\theta, \eta), W(\theta, \eta))^{\mathsf{T}} \\ &= \mathbf{X}(\theta) + \eta\mathbf{n}(\theta), \quad \mathbf{n}(\theta) = \frac{1}{|\partial\mathbf{X}/\partial\theta|}\begin{pmatrix} -\partial Y/\partial\theta \\ \partial X/\partial\theta \end{pmatrix}, \\ \Xi &= \{\boldsymbol{\xi} = (\theta, \eta) \mid \theta \in \mathbb{R}/(2\pi\mathbb{Z}),\ |\eta| < d_\Omega\}. \end{aligned} \tag{6.3}$$

The vector $\mathbf{n}$ is the unit normal on Γ. The image of $\eta = 0$ by the map $\mathbf{Z}$ is the curve Γ. This coordinate map can be extended to include $|\eta| = d_\Omega$. We compute the Jacobian and require that it be positive and bounded from above:

$$\det J_{\mathbf{Z}}(\boldsymbol{\xi}) = \left|\frac{\partial\mathbf{X}}{\partial\theta}\right| - \eta\kappa(\theta), \quad J_{\mathbf{Z}}(\boldsymbol{\xi}) = \frac{\partial\mathbf{Z}}{\partial\boldsymbol{\xi}}, \tag{6.4}$$

where $\kappa(\theta)$ is the curvature of Γ at $\mathbf{X}(\theta)$ and is therefore bounded so long as $\mathbf{X}$ is a C^2 function of θ.

We now let

$$d_\Omega = \min(d'_\Omega, d_\eta, d_{\mathbb{U}}), \quad d'_\Omega = \frac{1}{2}\frac{\min_\theta|\partial\mathbf{X}/\partial\theta|}{\max_\theta \kappa(\theta)}, \tag{6.5}$$

where $d_{\mathbb{U}}$, introduced earlier, is the distance between Γ and the boundary $\partial\mathbb{U}$ of the computational domain. With this definition of d_Ω, the map $\mathbf{Z}$ defines a local coordinate system on Ω. Within Ω, η may be identified with the signed-distance function $\tilde{\eta}$. We see that

$$0 < \frac{1}{2}\min_\theta\left|\frac{\partial\mathbf{X}}{\partial\theta}\right| \leq \det J_{\mathbf{Z}}(\boldsymbol{\xi}) \leq \frac{3}{2}\max_\theta\left|\frac{\partial\mathbf{X}}{\partial\theta}\right|. \tag{6.6}$$

We can also compute

$$|J_{\mathbf{Z}}(\boldsymbol{\xi})|_2 \leq \frac{3}{2}\max_\theta\left|\frac{\partial\mathbf{X}}{\partial\theta}\right| \equiv C_1, \quad |J_{\mathbf{Z}}^{-1}(\boldsymbol{\xi})|_2 \leq \left(\frac{1}{2}\min_\theta\left|\frac{\partial\mathbf{X}}{\partial\theta}\right|\right)^{-1} \equiv C_2, \tag{6.7}$$

where $|\cdot|_2$ denotes the matrix norm induced by the Euclidean distance. Note that C_1 and C_2 are constants that do not depend on $\boldsymbol{\xi}$.

We start with the following observation:

LEMMA 6.1 *Let w be as in Lemma* 4.3. *Then, for* $\mathbf{x} \in \mathbb{U}$,

$$\frac{1}{w(\mathbf{x}-\mathbf{X}(\theta))} \leq \begin{cases} \frac{1}{\sin(d_\Omega/(2\sqrt{2}))}\frac{d_\Omega}{|\mathbf{x}-\mathbf{X}(\theta)|} & \text{if } |\mathbf{x}-\mathbf{X}(\theta)| < d_\Omega, \\ \frac{1}{\sin(d_\Omega/(2\sqrt{2}))} & \text{if } |\mathbf{x}-\mathbf{X}(\theta)| \geq d_\Omega. \end{cases} \tag{6.8}$$

PROOF: Let $\mathbf{y} = (x', y')^{\mathsf{T}} = (x - X(\theta), y - Y(\theta))^{\mathsf{T}} = \mathbf{x} - \mathbf{X}(\theta)$. We first note that

$$w(\mathbf{y}) = \max(\sin(x'/2), \sin(y'/2)) \geq \sin(\max(x'/2, y'/2)) \equiv \sin(w_m). \tag{6.9}$$

w_m is bounded by

$$2w_m \equiv \max(x', y') \geq \min_{\psi}(\max(|\mathbf{y}|\cos(\psi), |\mathbf{y}|\sin(\psi))) = \frac{|\mathbf{y}|}{\sqrt{2}}. \tag{6.10}$$

Combined with a trivial bound, we have

$$\frac{|\mathbf{y}|}{2\sqrt{2}} \leq w_m \leq \frac{|\mathbf{y}|}{2}. \tag{6.11}$$

Suppose $|\mathbf{y}| \geq d_\Omega$. Equation (6.11) implies

$$w_m \geq \frac{d_\Omega}{2\sqrt{2}}. \tag{6.12}$$

By definition (6.5) of d_Ω, $\mathbf{X}(\theta)$ is at least a distance d_Ω away from $\partial\mathbb{U}$. Recalling that $\mathbb{U}$ has side 2π, $2w_m = \max(x', y') = \max(x - X(\theta), y - Y(\theta)) \leq 2\pi - d_\Omega$. Combined with (6.12),

$$\frac{d_\Omega}{2\sqrt{2}} \leq w_m \leq \pi - \frac{d_\Omega}{2}. \tag{6.13}$$

Thus, we have

$$w \geq \sin(w_m) \geq \sin\left(\frac{d_\Omega}{2\sqrt{2}}\right). \tag{6.14}$$

On the other hand, suppose $|\mathbf{y}| < d_\Omega$. Equation (6.11) implies

$$w_m < \frac{d_\Omega}{2} < \frac{\pi}{2} \tag{6.15}$$

where we used $d_\Omega < \pi$, as should be geometrically clear. We have

$$w \geq \sin(w_m) \geq \sin\left(\frac{|\mathbf{y}|}{2\sqrt{2}}\right) \geq \left(\frac{\sin(d_\Omega/(2\sqrt{2}))}{d_\Omega}\right)|\mathbf{y}|. \tag{6.16}$$

The first inequality follows from (6.15) and (6.11), whereas the second inequality follows from the convexity of the sine function for $(0, \pi)$. Inequalities (6.14) and (6.16) produce the desired result. □

To state the next lemma, we must introduce some notation. Take any point $\mathbf{x} = \mathbf{Z}(\boldsymbol{\xi}_0) = (Z(\theta_0, \eta_0), W(\theta_0, \eta_0))^{\mathsf{T}}$ in Ω. Denote the closest point from $\mathbf{x}$ on Γ as $\mathbf{X}_\perp = \mathbf{Z}(\theta_0, 0)$ (Figure 6.1). Draw an open disc of radius d_Ω centered at $\mathbf{X}_\perp$, which we shall call D_{θ_0}. Note that $D_{\theta_0} \subset \Omega$. Take any point $\mathbf{X}(\theta_1) = \mathbf{Z}(\boldsymbol{\xi}_1) = (Z(\theta_1, 0), W(\theta_1, 0)) \in D_{\theta_0}$.

LEMMA 6.2 *For* $\mathbf{x}$, $\mathbf{X}(\theta_1)$, $\boldsymbol{\xi}_0$, *and* $\boldsymbol{\xi}_1$ *introduced above, we have*

$$|\mathbf{x} - \mathbf{X}(\theta_1)| = |\mathbf{Z}(\boldsymbol{\xi}_0) - \mathbf{Z}(\boldsymbol{\xi}_1)| \geq C_{\mathbf{Z}} |\boldsymbol{\xi}_0 - \boldsymbol{\xi}_1| \tag{6.17}$$

where $C_{\mathbf{Z}}$ *does not depend on* $\mathbf{x}$, $\mathbf{X}(\theta_1)$, $\boldsymbol{\xi}_0$, *or* $\boldsymbol{\xi}_1$.

PROOF: Draw a straight line $\mathcal{L}$ between $\mathbf{x}$ and $\mathbf{X}(\theta_1)$. This line is wholly contained in $D_{\theta_0} \subset \Omega$ since a disc is convex. Thus the inverse image $\mathbf{Z}^{-1}(\mathcal{L})$ is contained in Ξ and is a curve that connects $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_1$. Let the length of $\mathbf{Z}^{-1}(\mathcal{L})$ be Λ and introduce an arc length coordinate σ:

$$\boldsymbol{\xi}(\sigma) = (\theta(\sigma), \eta(\sigma))^{\mathsf{T}}, \quad \boldsymbol{\xi}(0) = \boldsymbol{\xi}_0, \quad \boldsymbol{\xi}(\Lambda) = \boldsymbol{\xi}_1. \tag{6.18}$$

Since a straight line is shorter in length than a curve, we have

$$|\boldsymbol{\xi}_0 - \boldsymbol{\xi}_1| \leq \Lambda. \tag{6.19}$$

The line $\mathcal{L}$ can also be parametrized with σ, and its length $|\mathcal{L}|$ calculated in terms of an integral in σ:

$$|\mathcal{L}| = \int_0^\Lambda \left| \frac{d\mathbf{Z}(\boldsymbol{\xi}(\sigma))}{d\sigma} \right| d\sigma = \int_0^\Lambda \left| J_{\mathbf{Z}} \frac{d\boldsymbol{\xi}}{d\sigma} \right| d\sigma \tag{6.20}$$

where $J_{\mathbf{Z}} = \partial \mathbf{Z} / \partial \boldsymbol{\xi}$. We can obtain the following lower bound on the integrand":

$$\begin{aligned} \left| J_{\mathbf{Z}} \frac{d\boldsymbol{\xi}}{d\sigma} \right| \geq \min_{\mathbf{v} \neq \mathbf{0}} \frac{|J_{\mathbf{Z}} \mathbf{v}|}{|\mathbf{v}|} &= \min_{\mathbf{v} \neq \mathbf{0}} \frac{|\mathbf{v}|}{|J_{\mathbf{Z}}^{-1} \mathbf{v}|} \\ &= \frac{1}{\max_{\mathbf{v} \neq \mathbf{0}} \left(|J_{\mathbf{Z}}^{-1} \mathbf{v}| / |\mathbf{v}| \right)} = \frac{1}{|J_{\mathbf{Z}}^{-1}|_2} \geq C > 0 \end{aligned} \tag{6.21}$$

where we used $|d\boldsymbol{\xi}/d\sigma| = 1$ (σ is the arc length coordinate) in the first inequality and (6.7) in the second. Substituting the above back into the integral,

$$|\mathcal{L}| \geq C\Lambda. \tag{6.22}$$

Combine the above with (6.19) to obtain the desired inequality. □

Now we are ready to prove the following global error estimate:

THEOREM 6.3 *Suppose* $\mathbf{F}$ *is* C^1 *and* $\mathbf{X}$ *is* C^2 *as a function of* θ, *and* ϕ *is of moment order at least* 1. *Assume also that* $|\partial \mathbf{X} / \partial \theta|$ *never vanishes. For sufficiently small* h *and* $\Delta\theta$ *the following global error estimate is valid*:

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\mathbb{U})} \leq C(h + \Delta\theta)\big(\log(h^{-1}) + \log((\Delta\theta)^{-1})\big) \tag{6.23}$$

where C *is a constant that does not depend on* h *or* $\Delta\theta$.

PROOF: Suppose $\mathbf{x} \in \mathbb{U} \setminus \Omega$. Then, the distance from $\mathbf{x}$ to Γ is at least d_Ω. We may thus apply inequality (5.16):

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \leq C(h + \Delta\theta) \tag{6.24}$$

where the constant C depends only on d_Ω, $\mathbf{F}$, $\mathbf{X}$, and its derivatives, and not on $\mathbf{x} \in \mathbb{U} \setminus \Omega$, h, or $\Delta\theta$.

Next suppose that $\mathbf{x} \in \Omega$. Without loss of generality, we assume that the θ-coordinate that corresponds to $\mathbf{x}$ is $\theta = 0$. That is to say, $\mathbf{x} = \mathbf{Z}(\boldsymbol{\xi}_0) = \mathbf{Z}(0, \eta_0)$ for some $\eta_0 < d_\Omega$. We let $\mathbf{X}_\perp = \mathbf{Z}(0,0) = \mathbf{X}(0)$, the closest point on Γ from $\mathbf{x}$. We first write the difference between $\mathbf{u}$ and $\mathbf{u}_h$ as follows: Define the following subset of θ:

$$\mathcal{D} = \{\theta \mid |\mathbf{X}_\perp - \mathbf{X}(\theta)| < d_\Omega\}. \tag{6.25}$$

Then

$$\mathbf{u} - \mathbf{u}_h = \mathbf{I}^a + \mathbf{I}^b,$$

$$\begin{aligned} \mathbf{I}_a &= \int\limits_{\theta \notin \mathcal{D}} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - \sum_{\theta_m \notin \mathcal{D}} (S^*(\mathbf{X})G_{h,\mathbf{x}})(\theta_m)\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{I}_b &= \int\limits_{\mathcal{D}} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - \sum_{\theta_m \in \mathcal{D}} (S^*(\mathbf{X})G_{h,\mathbf{x}})(\theta_m)\mathbf{F}(\theta_m)\Delta\theta. \end{aligned} \tag{6.26}$$

We thus divided the difference into regions of θ where $|\mathbf{X}_\perp - \mathbf{X}(\theta)| < d_\Omega$ or not.

Denote by Γ_a the portion of Γ that corresponds to $\theta \notin \mathcal{D}$. $\mathbf{I}_a$ can be written as

$$\begin{aligned} \mathbf{I}_a &= \mathbf{v} - \mathbf{v}_h, \\ \mathbf{v} &= \int\limits_{\mathbf{X}(\theta) \in \Gamma_a} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta, \\ \mathbf{v}_h &= \sum_{\mathbf{X}(\theta_m) \in \Gamma_a} (S^*(\mathbf{X})G_{h,\mathbf{x}})(\theta_m)\mathbf{F}(\theta_m)\Delta\theta. \end{aligned} \tag{6.27}$$

Any point on Γ_a is at least a distance d_Ω away from the three points $\mathbf{Z}(0, d_\Omega)$, $\mathbf{Z}(0, -d_\Omega)$, and $\mathbf{X}_\perp = \mathbf{Z}(0,0)$, as can be seen from the definition of the region Ω. Elementary geometric arguments show that the distance between Γ_a and $\mathbf{x}$ is at least $(\sqrt{3}/2)d_\Omega$. We can thus apply Corollary 5.2 to find that

$$|\mathbf{I}_a| \leq C(h + \Delta\theta) \tag{6.28}$$

where the constant C does not depend on $\mathbf{x} \in \Omega$, h, or $\Delta\theta$.

The task that remains is to bound $\mathbf{I}^b$. Define the subset of θ as

$$\mathcal{V} = \{\theta \mid |\theta| \le \epsilon, \epsilon \equiv \max(\Delta\theta, (\sqrt{2} r_\phi h / C_{\mathbf{Z}}) + h)\} \tag{6.29}$$

where $C_{\mathbf{Z}}$ is the same $C_{\mathbf{Z}}$ that appears in Lemma 6.2. We write $\mathbf{I}^b$ as

$$\begin{aligned} \mathbf{I}_b &= \mathbf{I}_{b1} + \mathbf{I}_{b2}, \\ \mathbf{I}_{b1} &= \int_{\mathcal{D}\backslash\mathcal{V}} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - \sum_{\theta_m \in \mathcal{D}\backslash\mathcal{V}} (S^*(\mathbf{X})G_{h,\mathbf{x}})(\theta_m)\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{I}_{b2} &= \int_{\mathcal{V}} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - \sum_{\theta_m \in \mathcal{V}} (S^* G_{h,\mathbf{x}})(\mathbf{X}(\theta_m))\mathbf{F}(\theta_m)\Delta\theta. \end{aligned} \tag{6.30}$$

We start by deriving bounds for $\mathbf{I}_{b1}$. Write $\mathbf{I}_{b1}$ as

$$\begin{aligned} \mathbf{I}_{b1} &= \mathbf{I}_{b1}^{\int} + \mathbf{I}_{b1}^{\delta} + \mathbf{I}_{b1}^{G}, \\ \mathbf{I}_{b1}^{\int} &= \int_{\mathcal{D}\backslash\mathcal{V}} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - \sum_{\theta_m \in \mathcal{D}\backslash\mathcal{V}} G_{\mathbf{x}}(\mathbf{X}(\theta_m))\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{I}_{b1}^{\delta} &= \sum_{\theta_m \in \mathcal{D}\backslash\mathcal{V}} (G_{\mathbf{x}}(\mathbf{X}(\theta_m)) - (S^*(\mathbf{X})G_{\mathbf{x}}))(\theta_m))\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{I}_{b1}^{G} &= \sum_{\theta_m \in \mathcal{D}\backslash\mathcal{V}} (S^*(\mathbf{X})(G_{\mathbf{x}} - G_{h,\mathbf{x}}))(\theta_m)\mathbf{F}(\theta_m)\Delta\theta. \end{aligned} \tag{6.31}$$

We first introduce some notation. Let $\mathcal{D} \setminus \mathcal{V}$ be characterized by

$$\mathcal{D} \setminus \mathcal{V} = \{\theta \mid \theta^- < \theta < -\epsilon,\ \epsilon < \theta < \theta^+\}. \tag{6.32}$$

Consider the region $\epsilon < \theta < \theta^+$. We let the smallest and the largest θ_m that belong to this region to be θ_P and θ_Q, respectively. Define $\tilde{\theta}_m^+$ to be

$$\tilde{\theta}_m^+ = \epsilon + (m - P)\Delta\theta \quad \text{so that} \quad \tilde{\theta}_m^+ \le \theta_m < \tilde{\theta}_m^+ + \Delta\theta. \tag{6.33}$$

We start by estimating $\mathbf{I}_{b1}^{\int}$. Denote the portion of $\mathbf{I}_{b1}^{\int}$ where $\theta > 0$ as $\mathbf{I}_{b1}^{\int +}$. This can be written as

$$
\begin{aligned}
\mathbf{I}_{b1}^{\int +} &= \int_{\epsilon}^{\theta^+} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - \sum_{m=P}^{Q} G_{\mathbf{x}}(\mathbf{X}(\theta_m))\mathbf{F}(\theta_m)\Delta\theta \\
&= \mathbf{J}^0 + \sum_{m=P}^{Q} \mathbf{J}_m^1, \\
\mathbf{J}_m^1 &= \int_{\tilde{\theta}_m^+}^{\tilde{\theta}_{m+1}^+} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta - G_{\mathbf{x}}(\mathbf{X}(\theta_m))\mathbf{F}(\theta_m)\Delta\theta, \\
\mathbf{J}^0 &= \int_{\tilde{\theta}_{Q+1}^+}^{\theta^+} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta.
\end{aligned}
\tag{6.34}
$$

$\mathbf{J}^0 = (J_1^0, J_2^0)^{\mathsf{T}}$ can be bounded as follows: Suppose $\tilde{\theta}_{Q+1}^+ < \theta^+$. Then

$$
\begin{aligned}
|J_i^0| &\le \left| \int_{\tilde{\theta}_{Q+1}^+}^{\theta^+} \sum_{j=1}^{2} G_{\mathbf{x},ij}(\mathbf{X}(\theta)) F_j(\theta) d\theta \right| \\
&\le \Delta\theta \max_{\tilde{\theta}_{Q+1}^+ \le \theta \le \theta^+} \sum_{j=1}^{2} |G_{\mathbf{x},ij}(\mathbf{X}(\theta)) F_j(\theta)| \\
&\le C\Delta\theta \max_{\tilde{\theta}_{Q+1}^+ \le \theta \le \theta^+} \left(\log(|\mathbf{x} - \mathbf{X}(\theta)|^{-1}) + 1 \right)
\end{aligned}
\tag{6.35}
$$

where we used Lemma 4.1 in the last line. Note that $|\mathbf{x} - \mathbf{X}(\theta^+)| \ge (\sqrt{3}/2)d_\Omega$ by the argument preceding equation (6.28). Thus,

$$
\max_{\tilde{\theta}_{Q+1}^+ \le \theta \le \theta^+} |\mathbf{x} - \mathbf{X}(\theta)| \ge \frac{\sqrt{3}}{2} d_\Omega - \max_{\theta} \left| \frac{\partial \mathbf{X}}{\partial \theta} \right| \Delta\theta \ge \frac{d_\Omega}{2}. \tag{6.36}
$$

We used $|\tilde{\theta}_{Q+1}^+ - \theta^+| \le \Delta\theta$. The last inequality above holds for sufficiently small $\Delta\theta$. This shows that

$$
|J_i^0| \le C\Delta\theta. \tag{6.37}
$$

The $\tilde{\theta}_{Q+1}^+ > \theta^+$ case follows similarly.

We now turn to $\mathbf{J}_m^1 = (J_{m,1}^1, J_{m,2}^1)^\mathsf{T}$.

$$
\begin{aligned}
|J_{m,i}^1| &\le (\Delta\theta)^2 \max_{\tilde\theta_m^+ \le \theta \le \tilde\theta_{m+1}^+} \left| \sum_{j=1}^2 \frac{\partial}{\partial\theta} \big(G_{\mathbf{x},ij}(\mathbf{X}(\theta)) F_j(\theta)\big) \right| \\
&\le (\Delta\theta)^2 \sum_{j=1}^2 \max_{\tilde\theta_m^+ \le \theta \le \tilde\theta_{m+1}^+} \left(\left| \frac{\partial G_{\mathbf{x},ij}(\mathbf{X}(\theta))}{\partial\theta} \right| |F_j(\theta)| \right. \\
&\qquad \left. + |G_{\mathbf{x},ij}(\mathbf{X}(\theta))| \left| \frac{\partial F_j(\theta)}{\partial\theta} \right| \right).
\end{aligned}
\tag{6.38}
$$

From Lemma 4.1, we have

$$
\begin{aligned}
\left| \frac{\partial G_{\mathbf{x},ij}(\mathbf{X}(\theta))}{\partial\theta} \right| &\le C(|\mathbf{x} - \mathbf{X}(\theta)|^{-1} + 1) \left| \frac{\partial \mathbf{X}(\theta)}{\partial\theta} \right|, \\
|G_{\mathbf{x},ij}(\mathbf{X}(\theta))| &\le C \left(\log(|\mathbf{x} - \mathbf{X}(\theta)|^{-1}) + 1 \right).
\end{aligned}
\tag{6.39}
$$

We thus have

$$
\begin{aligned}
|J_{m,i}^1| &\le C(\Delta\theta)^2 \max_{\tilde\theta_m^+ \le \theta \le \tilde\theta_{m+1}^+} \left(|\mathbf{x} - \mathbf{X}(\theta)|^{-1} + \log(|\mathbf{x} - \mathbf{X}(\theta)|^{-1}) + 1 \right) \\
&\le C(\Delta\theta)^2 \max_{\tilde\theta_m^+ \le \theta \le \tilde\theta_{m+1}^+} |\mathbf{x} - \mathbf{X}(\theta)|^{-1}
\end{aligned}
\tag{6.40}
$$

where C does not depend on $\mathbf{x} \in \Omega$, h, or $\Delta\theta$. In the last inequality above, we discarded the logarithmic and constant terms since they can be dominated by $|\mathbf{x} - \mathbf{X}(\theta)|^{-1}$ for $|\mathbf{x} - \mathbf{X}(\theta)| < d_\Omega$ by taking the constant C in the last line sufficiently large:

$$
\begin{aligned}
\sum_{m=P}^{Q} |J_{m,i}^1| &\le C(\Delta\theta)^2 \sum_{m=P}^{Q} \max_{\tilde\theta_m^+ \le \theta \le \tilde\theta_{m+1}^+} \frac{1}{|\mathbf{x} - \mathbf{X}(\theta)|} \\
&\le C(\Delta\theta)^2 \sum_{m=P}^{Q} \max_{\tilde\theta_m^+ \le \theta \le \tilde\theta_{m+1}^+} \frac{1}{C_{\mathbf{Z}} \sqrt{\theta^2 + \eta_0^2}} \\
&\le C\Delta\theta \sum_{m=P}^{Q} \frac{\Delta\theta}{\sqrt{(\tilde\theta_m^+)^2 + \eta_0^2}}
\end{aligned}
\tag{6.41}
$$

where we used Lemma 6.2 in the second inequality.

We now estimate the last sum:

$$\sum_{m=P}^{Q} \frac{\Delta\theta}{\sqrt{(\tilde{\theta}_m^+)^2+\eta_0^2}} \leq \sum_{m=0}^{Q-P} \frac{\Delta\theta}{\epsilon+m\Delta\theta} \leq \frac{\Delta\theta}{\epsilon} + \int_0^{\pi-\epsilon} \frac{dr}{\epsilon+r} = \frac{\Delta\theta}{\epsilon} + \log\left(\frac{\pi}{\epsilon}\right) \leq C(1+\log(\epsilon^{-1})) \tag{6.42}$$

where we used $\epsilon + (Q-P)\Delta\theta \leq \theta^+ \leq \pi$ in the second inequality and $\epsilon \geq \Delta\theta$ (see equation (6.29)) in the last inequality.

Combining (6.37), (6.41), and (6.42) to estimate the i^{th} component of $\mathbf{I}_{b1}^{\int+}$, we get

$$\left|I_{b1,i}^{\int+}\right| = \left|J_i^0 + \sum_{m=P}^{Q} J_{m,i}^1\right| \leq |J_i^0| + \sum_{m=P}^{Q} |J_{m,i}^1| \leq C\Delta\theta + C\Delta\theta(1+\log(\epsilon^{-1})). \tag{6.43}$$

We can obtain the same bound for the $\theta < 0$ part of $\mathbf{I}_{b1}^{\int}$, and we have the estimate

$$\left|\mathbf{I}_{b1}^{\int}\right| \leq C\Delta\theta(1+\log(\epsilon^{-1})) \tag{6.44}$$

where C does not depend on $\mathbf{x} \in \Omega$, h, or $\Delta\theta$.

We turn next to $\mathbf{I}_{b1}^{\delta}$. First consider the difference in the sum:

$$\begin{aligned} &|G_{\mathbf{x},ij}(\mathbf{X}(\theta)) - (S^*(\mathbf{X})G_{\mathbf{x},ij})(\theta)| \\ &\leq \max_{\alpha+\beta=1} \Big(\max_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta)}} |\partial_x^\alpha \partial_y^\beta G_{\mathbf{x}}(\mathbf{y})|\Big) h \\ &\leq \max_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta)}} \frac{Ch}{|\mathbf{x}-\mathbf{y}|} = \frac{Ch}{\min_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta)}} |\mathbf{x}-\mathbf{y}|} \end{aligned} \tag{6.45}$$

where we used Lemma 3.2 in the first inequality and Lemma 4.1 in the second. We obtain a lower bound on the denominator of the last expression:

$$\begin{aligned} \min_{\mathbf{y}\in\mathcal{R}_{\mathbf{X}(\theta)}} |\mathbf{x}-\mathbf{y}| &\geq |\mathbf{x}-\mathbf{X}(\theta)| - \sqrt{2}\, r_\phi h \\ &\geq C_{\mathbf{Z}}\sqrt{\theta_m^2+\eta_0^2} - \sqrt{2}\, r_\phi h \\ &\geq C_{\mathbf{Z}}\left(|\theta_m| - \frac{\sqrt{2}\, r_\phi h}{C_{\mathbf{Z}}}\right) \geq C_{\mathbf{Z}}\left(|\theta_m| - (\epsilon - h)\right). \end{aligned} \tag{6.46}$$

In the first inequality, we used the fact that $\mathcal{R}_{\mathbf{X}(\theta)}$ is contained in a closed disc centered at $\mathbf{X}(\theta)$ of radius $\sqrt{2}\, r_\phi h$. We used Lemma 6.2 in the second inequality.

In the last inequality, we used $\epsilon - h \geq \sqrt{2} r_\phi h / C_{\mathbf{Z}}$ as follows from the definition of ϵ, (6.29). Note that the last expression is greater than 0 since $|\theta_m| > \epsilon$.

Combining (6.45) and (6.46), we can estimate $\mathbf{I}^\delta_{b1} = (I^\delta_{b1,1}, I^\delta_{b1,2})^\mathsf{T}$:

$$
\begin{aligned}
|I^\delta_{b1,i}| &\leq \sum_{\theta_m \in \mathcal{D} \setminus \mathcal{V}} \sum_{j=1}^{2} |G_{\mathbf{x},ij}(\mathbf{X}(\theta_m)) - (S^*(\mathbf{X}) G_{\mathbf{x},ij})(\theta_m)| \, |F_j(\theta_m)| \Delta\theta \\
&\leq \sum_{\theta_m \in \mathcal{D} \setminus \mathcal{V}} \sum_{j=1}^{2} \frac{Ch |F_j(\theta_m)| \Delta\theta}{C_{\mathbf{Z}} (|\theta_m| - (\epsilon - h))} \leq Ch \sum_{\theta_m \in \mathcal{D} \setminus \mathcal{V}} \frac{\Delta\theta}{|\theta_m| - (\epsilon - h)}.
\end{aligned}
\tag{6.47}
$$

Consider the $\theta_m > 0$ part of the last sum.

$$
\begin{aligned}
\sum_{m=P}^{Q} \frac{\Delta\theta}{\theta_m - (\epsilon - h)} &\leq \frac{\Delta\theta}{\theta_P - (\epsilon - h)} + \int_{\theta_P}^{\theta_{Q+1}} \frac{d\theta}{\theta - (\epsilon - h)} \\
&\leq \frac{\Delta\theta}{h} + \int_{\epsilon}^{\pi} \frac{d\theta}{\theta - (\epsilon - h)} \leq C \left(\frac{\Delta\theta}{h} + \log(h^{-1}) \right).
\end{aligned}
\tag{6.48}
$$

We obtain the same estimate for $\theta_m < 0$ part of the sum. Substituting this back into (6.47),

$$
|\mathbf{I}^\delta_{b1}| \leq C(\Delta\theta + h \log(h^{-1}))
\tag{6.49}
$$

where C is a constant that does not depend on $\mathbf{x} \in \Omega$, h, or $\Delta\theta$.

We now estimate $\mathbf{I}^G_{b1}$. We estimate the difference in the sum:

$$
\begin{aligned}
|S^*(\mathbf{X})(G_{\mathbf{x},ij} - G_{h,\mathbf{x},ij})(\theta)| &\leq C \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta)}} |G_{\mathbf{x},ij}(\mathbf{y}) - G_{h,\mathbf{x},ij}(\mathbf{y})| \\
&\leq \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta)}} \frac{Ch}{w(\mathbf{x} - \mathbf{y})} \leq \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta)}} \frac{Ch}{|\mathbf{x} - \mathbf{y}|}
\end{aligned}
\tag{6.50}
$$

where we used Lemma 3.1 in the first inequality, Proposition 4.5 in the second, and Lemma 6.1 in the third. This estimate has the same form as equation (6.45) and thus allows us to proceed in the same way as when we bounded $\mathbf{I}^\delta_{b1}$:

$$
|\mathbf{I}^G_{b1}| \leq C(\Delta\theta + h \log(h^{-1}))
\tag{6.51}
$$

where C is a constant that does not depend on $\mathbf{x} \in \Omega$, h, or $\Delta\theta$.

Combining (6.44), (6.49), and (6.51) we have

$$
|\mathbf{I}_{b1}| \leq C(\Delta\theta + \Delta\theta \log(\epsilon^{-1}) + h \log(h^{-1}))
\tag{6.52}
$$

where C is a constant that does not depend on $\mathbf{x} \in \Omega$, h, or $\Delta\theta$.

Finally, we must bound $\mathbf{I}_{b2}$. We separate this as follows:

$$\mathbf{I}_{b2} = \mathbf{I}_{b2}^{\int} + \mathbf{I}_{b2}^{\sum},$$
$$\mathbf{I}_{b2}^{\int} = \int_{\mathcal{V}} G_{\mathbf{x}}(\mathbf{X}(\theta))\mathbf{F}(\theta)d\theta, \tag{6.53}$$
$$\mathbf{I}_{b2}^{\sum} = -\sum_{\theta_m\in\mathcal{V}} (S^*(\mathbf{X})G_{h,\mathbf{x}})(\theta_m)\mathbf{F}(\theta_m)\Delta\theta.$$

First, we bound $\mathbf{I}_{b2}^{\int} = (I_{b2,1}^{\int}, I_{b2,2}^{\int})^{\mathsf{T}}$:

$$\begin{aligned}\left|I_{b2,i}^{\int}\right| &\le \int_{-\epsilon}^{\epsilon}\sum_{j=1}^{2}|G_{\mathbf{x},ij}(\mathbf{X}(\theta))|\,|F_j(\theta)|d\theta\\ &\le C\int_{-\epsilon}^{\epsilon}\left(\log\left(|\mathbf{x}-\mathbf{X}(\theta)|^{-1}\right)+1\right)d\theta\\ &\le C\int_{-\epsilon}^{\epsilon}(\log(1/|\theta|)+1)d\theta \le C\epsilon(1+\log(\epsilon^{-1}))\end{aligned} \tag{6.54}$$

where we used Lemma 4.1 in the first inequality. In the second inequality, we used Lemma 6.2 from which we see that $|\mathbf{x}-\mathbf{X}(\theta)| \ge C_{\mathbf{Z}}\sqrt{\theta^2+\eta_0^2} \ge C_{\mathbf{Z}}|\theta|$.

Next we bound $\mathbf{I}_{b2}^{\sum} = (I_{b2,1}^{\sum}, I_{b2,2}^{\sum})^{\mathsf{T}}$. This yields

$$\begin{aligned}\left|I_{b2,i}^{\sum}\right| &\le \sum_{\theta_m\in\mathcal{V}}\sum_{j=1}^{2}|(S^*(\mathbf{X})G_{h,\mathbf{x},ij})(\theta_m)|\,|F_j(\theta_m)|\Delta\theta\\ &\le \sum_{\theta_m\in\mathcal{V}} C\log(h^{-1})\Delta\theta\end{aligned} \tag{6.55}$$

where we used Lemma 3.1 and Lemma 4.2 in the last inequality. Since $\mathcal{V}$ spans $-\epsilon \le \theta \le \epsilon$, the number of terms in the last sum above is at most $1+2\epsilon/\Delta\theta$. Thus

$$\left|I_{b2,i}^{\sum}\right| \le C(\Delta\theta + 2\epsilon\log(h^{-1})). \tag{6.56}$$

Combining (6.55) and (6.56), we have

$$|\mathbf{I}_{b2}| \le C\left(\Delta\theta+\epsilon+\epsilon\log(\epsilon^{-1})+\epsilon\log(h^{-1})\right) \tag{6.57}$$

where C is a constant that does not depend on $\mathbf{x}\in\Omega$, h, or $\Delta\theta$.

We now only have to collect results. From estimates (6.28), (6.52), and (6.57) we can estimate the difference (6.26):

$$\begin{aligned}|\mathbf{u}(\mathbf{x})-\mathbf{u}_h(\mathbf{x})| &\le |\mathbf{I}_a|+|\mathbf{I}_{b1}|+|\mathbf{I}_{b2}|\\ &\le C(h+\Delta\theta)\left(1+\log(h^{-1})+\log((\Delta\theta)^{-1})\right)\end{aligned} \tag{6.58}$$

where $\mathbf{x} \in \Omega$ and the constant C does not depend on $\mathbf{x}$, h, or $\Delta\theta$. We used $\epsilon \le C(h + \Delta\theta)$ and $\log(\epsilon^{-1}) \le C(\log(h^{-1}) + \log((\Delta\theta)^{-1}))$ as follows from the definition of ϵ in (6.29). We can combine the above with the estimate (6.24) valid for $\mathbf{x} \notin \Omega$ to obtain

$$|\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \le C(h + \Delta\theta)\big(\log(h^{-1}) + \log((\Delta\theta)^{-1})\big) \tag{6.59}$$

where $\mathbf{x} \in \mathbb{U}$ and the constant C does not depend on $\mathbf{x}$, h, or $\Delta\theta$. We have used the fact that a constant term is dominated by $\log(h^{-1})$ or $\log((\Delta\theta)^{-1})$ for small enough h or $\Delta\theta$. We can now take the supremum on the left-hand side of the above to obtain the desired estimate. □

We can deduce some interesting conclusions from the above theorem. Refine $\Delta\theta$ proportionally to h^α where α is positive. According to the above theorem:

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\mathbb{U})} \le C(h + h^\alpha)\log(h^{-1}). \tag{6.60}$$

This tells us that the immersed boundary solution converges to the true solution everywhere for any $\alpha > 0$. Any $\alpha < 1$ would lead to suboptimal convergence. If $\alpha > 1$, the asymptotic rate of convergence would be the same as if $\alpha = 1$, but would be a waste of computational resources since $\alpha > 1$ means more immersed boundary points compared to the $\alpha = 1$ case. We may combine these observations into a corollary.

COROLLARY 6.4 *Refine $\Delta\theta$ proportionally to h^α. The immersed boundary solution $\mathbf{u}_h$ converges to the true solution $\mathbf{u}$ everywhere in the domain $\mathbb{U}$ for any $\alpha > 0$. If in particular $\Delta\theta$ is refined proportionally to h, we have the error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(\mathbb{U})} \le C h \log(h^{-1}) \tag{6.61}$$

where the constant C does not depend on h.

7 A Simple Dynamic Problem

As an application of the foregoing estimates, we consider a small-amplitude dynamic problem. We let

$$\Delta\mathbf{u}(\mathbf{x}, t) = \nabla p(\mathbf{x}, t) - \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x}, t), \quad \nabla\cdot\mathbf{u}(\mathbf{x}, t) = 0, \tag{7.1}$$

$$\mathbf{f}(\mathbf{x}, t) = \int_{-\pi}^{\pi} \mathbf{F}(\theta, t)\delta(\mathbf{x} - \mathbf{X}_0(\theta))d\theta, \quad \mathbf{F}(\theta, t) = -K(\mathbf{X}(\theta, t) - \mathbf{X}_0(\theta)), \tag{7.2}$$

$$\frac{\partial\mathbf{X}(\theta, t)}{\partial t} = \mathbf{u}(\mathbf{X}_0(\theta), t) = \int_{\mathbb{U}} \mathbf{u}(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{X}_0(\theta))d\mathbf{x}. \tag{7.3}$$

The force law we consider is a simple restoring force to the fixed locations $\mathbf{X}_0$. Assuming that the immersed boundary points do not deviate significantly from the locations $\mathbf{X}_0$, we spread the forces at fixed locations $\mathbf{X}_0$ and interpolate the velocity at the same fixed locations. This is analogous to the linear approximation made in the theory of small-amplitude water waves [35]. In the form of an integral, the above system can be written as

$$
\begin{aligned}
\frac{\partial \mathbf{X}}{\partial t}(\theta, t) &= (\mathcal{L}\mathbf{F})(\theta, t) \equiv \mathbf{u}(\mathbf{X}_0(\theta), t), \\
\mathbf{u}(\mathbf{x}, t) &= (\mathcal{Q}\mathbf{F})(\mathbf{x}, t) \equiv \int_{-\pi}^{\pi} G(\mathbf{x} - \mathbf{X}_0(\theta))\mathbf{F}(\theta, t)\, d\theta, \\
\mathbf{F}(\theta, t) &= -K(\mathbf{X}(\theta, t) - \mathbf{X}_0(\theta)).
\end{aligned} \tag{7.4}
$$

A discretization of the above using the immersed boundary method is the following:

$$ L_h \mathbf{u}_h^n(\mathbf{x}) = \mathbf{D}_h p_h^n(\mathbf{x}) - \mathbf{f}_h^n(\mathbf{x}) + \mathbf{g}_h^n(\mathbf{x}), \quad \mathbf{D}_h \cdot \mathbf{u}_h^n(\mathbf{x}) = 0, \tag{7.5} $$

$$ \mathbf{f}_h^n(\mathbf{x}) = (S(\mathbf{X}_0)\mathbf{F}_h^n)(\mathbf{x}), \quad \mathbf{F}_h^n(\theta_m) = -K(\mathbf{X}_h^n(\theta_m) - \mathbf{X}_0(\theta_m)), \tag{7.6} $$

$$ D_t^- \mathbf{X}_h^n(\theta_m) = (S^*(\mathbf{X}_0)\mathbf{u}_h^n)(\theta_m), \quad D_t^- \mathbf{X}_h^n(\theta_m) \equiv \frac{\mathbf{X}_h^n(\theta_m) - \mathbf{X}_h^{n-1}(\theta_m)}{\Delta t}. \tag{7.7} $$

The superscript n refers to the immersed boundary approximation at time $t = n\Delta t$. $S(\mathbf{X}_0)$ and $S^*(\mathbf{X}_0)$ denote the spreading and interpolation operators, respectively, performed at the points $\mathbf{X}_0(\theta_m)$. We have used a backward Euler discretization for the time derivative.

The above can be written in the following form, which parallels (7.4):

$$
\begin{aligned}
D_t^- \mathbf{X}_h^n(\theta_m) &= (\mathcal{L}_h \mathbf{F}_h^n)(\theta_m) \equiv (S^*(\mathbf{X}_0)\mathbf{u}_h^n)(\theta_m), \\
\mathbf{u}_h^n(\mathbf{x}) &= (\mathcal{Q}_h \mathbf{F}_h^n)(\mathbf{x}) \equiv \sum_{\mathbf{y} \in \mathcal{G}_h} G_h(\mathbf{x} - \mathbf{y})(S(\mathbf{X}_0)\mathbf{F}_h^n)(\mathbf{y}), \\
\mathbf{F}_h^n(\theta_m) &= -K(\mathbf{X}_h^n(\theta_m) - \mathbf{X}_0(\theta_m)).
\end{aligned} \tag{7.8}
$$

For simplicity, in this section, we shall assume that $\Delta\theta$ is refined proportionally to h. We start with the following lemma:

LEMMA 7.1 *Suppose* $(\mathcal{Q}\mathbf{F})(\mathbf{x})$ *is a Lipschitz-continuous function in* $\mathbf{x}$. *Then*

$$ \max_m |(\mathcal{L}\mathbf{F})(\theta_m) - (\mathcal{L}_h \mathbf{F})(\theta_m)| \le C h \log(h^{-1}) \tag{7.9} $$

where C *is a constant independent of* h *and* $\Delta\theta$.

PROOF: Let $\mathbf{v}(\mathbf{x}) = (\mathcal{Q}\mathbf{F})(\mathbf{x})$ and $\mathbf{v}_h(\mathbf{x}) = (\mathcal{Q}_h\mathbf{F})(\mathbf{x})$. We have

$$\begin{aligned} |(\mathcal{L}\mathbf{F})(\theta_m) - (\mathcal{L}_h\mathbf{F})(\theta_m)| &\le |\mathbf{v}(\mathbf{X}_0(\theta_m)) - (S^*(\mathbf{X}_0)\mathbf{v}_h)(\theta_m)| \\ &\le E_1 + E_2, \\ E_1 &= |\mathbf{v}(\mathbf{X}_0(\theta_m)) - (S^*(\mathbf{X}_0)\mathbf{v})(\theta_m)|, \\ E_2 &= |(S^*(\mathbf{X}_0)(\mathbf{v}(\mathbf{x}) - \mathbf{v}_h(\mathbf{x})))(\theta_m)|. \end{aligned} \tag{7.10}$$

Given that $\mathbf{v}(\mathbf{x}) = (\mathcal{Q}\mathbf{F})(\mathbf{x})$ is Lipschitz-continuous and ϕ is of moment order at least 1,

$$E_1 = |\mathbf{v}(\mathbf{X}_0(\theta_m)) - (S^*(\mathbf{X}_0)\mathbf{v})(\theta_m)| \le Ch \tag{7.11}$$

where we used (3.18). On the other hand,

$$E_2 = |(S^*(\mathbf{X}_0)(\mathbf{v}(\mathbf{x}) - \mathbf{v}_h(\mathbf{x})))(\theta_m)| \le C \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0(\theta_m)}} |\mathbf{v}(\mathbf{x}) - \mathbf{v}_h(\mathbf{x})| \tag{7.12}$$

where we used Lemma 3.1 in the inequality above. We know from Corollary 6.4 that

$$\|\mathbf{v} - \mathbf{v}_h\|_{L^\infty(\mathbb{U})} = \|(\mathcal{Q}\mathbf{F}) - (\mathcal{Q}_h\mathbf{F})\|_{L^\infty(\mathbb{U})} \le Ch\log(h^{-1}). \tag{7.13}$$

Therefore,

$$E_2 \le C \max_{\mathbf{x}\in\mathcal{R}_{\mathbf{X}_0(\theta_m)}} |\mathbf{v}(\mathbf{x}) - \mathbf{v}_h(\mathbf{x})| \le C\|\mathbf{v} - \mathbf{v}_h\|_{L^\infty(\mathbb{U})} \le Ch\log(h^{-1}). \tag{7.14}$$

Combining (7.11) and (7.14), we obtain the desired result. □

We note that the velocity field $\mathbf{u} = \mathcal{Q}\mathbf{F}$ should be Lipschitz-continuous in $\mathbf{x}$ if $\mathbf{F}$ and $\mathbf{X}_0$ are smooth enough. After all, if $\mathbf{u}$ is a classical solution to the Stokes problem, $\mathbf{u}$ is continuous at the immersed boundary and its derivatives satisfy jump conditions whose values are given in terms of $\mathbf{F}$ and $\mathbf{X}_0$ [26]. Note also that the above lemma is of independent interest. It states that the interpolated velocity field using S^* converges to the true solution.

We now turn to the next lemma.

LEMMA 7.2 *The matrix $\mathcal{L}_h$ is symmetric positive semidefinite.*

PROOF: The operator $\mathcal{L}_h$ acting on $\mathbf{F}$ can be written as

$$\mathcal{L}_h\mathbf{F} = S^*(\mathbf{X}_0)\sum_{\mathbf{y}\in\mathcal{G}_h} G_h(\mathbf{x}-\mathbf{y})(S(\mathbf{X}_0)\mathbf{F})(\mathbf{y}). \tag{7.15}$$

Since G_h is a symmetric positive semidefinite matrix (see equation (2.17)) and $S(\mathbf{X}_0)$ and $S^*(\mathbf{X}_0)$ are transposes of each other, it follows that $\mathcal{L}_h$ is symmetric positive semidefinite. □

Define the discrete L^2 norm for functions defined on the immersed boundary grid:

$$\|\mathbf{X}\|_{L^2_\Delta(\Theta)} = \left(\sum_{m=1}^{M} |\mathbf{X}(\theta_m)|^2 \Delta\theta\right)^{1/2}. \tag{7.16}$$

We can now prove the following theorem:

THEOREM 7.3 *Suppose* $\mathbf{X}$ *is a smooth function in* t *and that* θ *and* $\mathbf{X}_0$ *satisfy the conditions of Theorem* 6.3. *Then the difference between* $\mathbf{X}$ *and the immersed boundary approximation* $\mathbf{X}_h$ *satisfies the following bound*:

$$\|\mathbf{X}(\cdot, T) - \mathbf{X}_h^n\|_{L^2_\Delta(\Theta)} \le CT(\Delta t + h\log(h^{-1})), \quad T = n\Delta t, \tag{7.17}$$

where the constant C *does not depend on* Δt *or* h.

PROOF: The idea of the proof is that Lemma 7.1 gives us consistency and Lemma 7.2 gives us the requisite stability.

Let $\mathbf{X}^n(\theta) = \mathbf{X}(\theta, n\Delta t)$. We first observe

$$(D_t^- \mathbf{X}^n)(\theta_m) = \frac{\partial \mathbf{X}}{\partial t}(\theta_m, n\Delta t) + \mathbf{J}_t^n(\theta_m), \quad |\mathbf{J}_t^n(\theta_m)| \le C\Delta t, \tag{7.18}$$

where C does not depend on m, n, h, or Δt. Next, we see that

$$(\mathcal{L}_h K(\mathbf{X}^n - \mathbf{X}_0))(\theta_m) = (\mathcal{L}K(\mathbf{X}^n - \mathbf{X}_0))(\theta_m) + \mathbf{J}_h^n(\theta_m), \qquad |\mathbf{J}_h^n(\theta_m)| \le Ch\log(h^{-1}), \tag{7.19}$$

where we used Lemma 7.1. The constant C above does not depend on m, n, h, or Δt. Noting that

$$\frac{\partial \mathbf{X}}{\partial t}(\theta_m, n\Delta t) = -(\mathcal{L}K(\mathbf{X}^n - \mathbf{X}_0))(\theta_m), \tag{7.20}$$

we immediately conclude that

$$(D_t^- \mathbf{X}^n)(\theta_m) = -(\mathcal{L}_h K(\mathbf{X}^n - \mathbf{X}_0))(\theta_m) + \mathbf{J}_t^n(\theta_m) + \mathbf{J}_h^n(\theta_m). \tag{7.21}$$

Since $\mathbf{X}_h^n$ satisfies the above discrete evolution equation exactly, we have

$$\begin{gathered}(D_t^- \mathbf{E}^n)(\theta_m) = -K(\mathcal{L}_h \mathbf{E}^n)(\theta_m)) + \mathbf{J}_t^n(\theta_m) + \mathbf{J}_h^n(\theta_m), \\ \mathbf{E}^n \equiv \mathbf{X}^n - \mathbf{X}_h^n, \quad \|\mathbf{E}^0\|_{L^2_\Delta(\Theta)} = 0.\end{gathered} \tag{7.22}$$

Define the discrete θ inner product:

$$(\mathbf{U}, \mathbf{V})_\theta = \sum_{m=1}^{M} (\mathbf{U}(\theta_m), \mathbf{V}(\theta_m))\Delta\theta \tag{7.23}$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product. Taking the discrete θ inner product on both sides of equation (7.22),

$$(\mathbf{E}^n, D_t^- \mathbf{E}^n)_\theta = -K(\mathbf{E}^n, \mathcal{L}_h \mathbf{E}^n)_\theta + (\mathbf{E}^n, \mathbf{J}_t^n + \mathbf{J}_h^n)_\theta. \tag{7.24}$$

Using Lemma 7.2, we see that the first term on the right-hand side is nonpositive. Dropping this term and substituting the definition of D_t^-, we find

$$\|\mathbf{E}^n\|^2_{L^2_\Delta(\Theta)} - (\mathbf{E}^n, \mathbf{E}^{n-1})_\theta \le \Delta t(\mathbf{E}^n, \mathbf{J}_t^n + \mathbf{J}_h^n)_\theta. \tag{7.25}$$

Using the Cauchy-Schwartz inequality and dividing both sides by $\|\mathbf{E}^n\|_{L^2_\Delta(\Theta)}$,

$$\begin{aligned}\|\mathbf{E}^n\|_{L^2_\Delta(\Theta)} - \|\mathbf{E}^{n-1}\|_{L^2_\Delta(\Theta)} &\leq \Delta t \|\mathbf{J}^n_t + \mathbf{J}^n_h\|_{L^2_\Delta(\Theta)} \\ &\leq \Delta t C(\Delta t + h\log(h^{-1})).\end{aligned} \tag{7.26}$$

Since $\|\mathbf{E}^0\|_{L^2_\Delta(\Theta)} = 0$, we immediately conclude that

$$\|\mathbf{E}^n\|_{L^2_\Delta(\Theta)} \leq TC(\Delta t + h\log(h^{-1})). \tag{7.27}$$

□

8 Computational Demonstration

We take the stationary model problem for which we proved convergence and perform computational experiments.

We shall examine the following six different discrete delta functions. We mainly follow [11] in the choice of discrete delta functions used to examine convergence. The simplest one we use is the following:

$$\phi^{ch}(r) = \begin{cases} \frac{1}{2}, & -1 \leq r < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

This function is discontinuous and never used in practice, but is nonetheless included here as an example of a discrete delta function that satisfies only one moment condition.

The next function we consider is the hat function:

$$\phi^H(r) = \begin{cases} \min(r+1, r-1), & |r| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

We shall also use the wider hat function:

$$\phi^{wH}(r) = \begin{cases} \frac{1}{4}\min(r+2, r-2), & |r| \leq 2, \\ 0 & \text{otherwise.} \end{cases} \tag{8.3}$$

When the support of the delta function is 4, one can accommodate up to four moment conditions [36], and the following is one such function [21, 36, 38]:

$$\phi^c(r) = \begin{cases} 1 - |r| - |r|^2 + \frac{1}{2}|r|^3, & |r| < 1, \\ 1 - \frac{11}{6}|r| + |r|^2 - \frac{1}{6}|r|^3, & 1 < |r| \leq 2, \\ 0. & 2 < |r|. \end{cases} \tag{8.4}$$

In [25], ϕ is derived systematically by imposing certain conditions to be satisfied by ϕ including the conditions we discussed in Section 3. Two functions that satisfy these conditions to varying degrees are the functions ϕ^{IB} and ϕ^{IB6}. ϕ^{IB} is defined as

$$\phi^{IB}(r) = \begin{cases} \frac{1}{8}(3 - 2|r| + \sqrt{1 + 4|r| - 4r^2}), & |r| \leq 1, \\ \frac{1}{8}(5 - 2|r| - \sqrt{-7 + 12|r| - 4r^2}), & 1 < |r| \leq 2, \\ 0, & 2 < |r|. \end{cases} \tag{8.5}$$

ϕ^{IB6} first appeared in [32]:

$$\phi^{IB6}(r) = \begin{cases} \phi_1^{IB6}(r), & |r| \le 1, \\ \frac{21}{16} + \frac{7}{12}|r| - \frac{7}{8}|r|^2 + \frac{1}{6}|r|^3 - \frac{3}{2}\phi_1^{IB6}(|r|-1), & 1 < |r| \le 2, \\ \frac{9}{8} - \frac{23}{12}|r| + \frac{3}{4}|r|^2 - \frac{1}{12}|r|^3 + \frac{1}{2}\phi_1^{IB6}(|r|-2), & 2 < |r| \le 3, \\ 0, & 3 < |r|. \end{cases} \tag{8.6}$$

$$\phi_1^{IB6}(r) = \frac{61}{112} - \frac{11}{42}|r| - \frac{11}{56}|r|^2 \frac{1}{12}|r|^3 + \frac{\sqrt{3}}{336}\big(243 + 1584|r| - 748|r|^2 - 1560|r|^3 + 500|r|^4 + 336|r|^5 - 112|r|^6\big)^{\frac{1}{2}}.$$

We let $N = 128 \times 2^k$, $k = 0, 1, 2, 3$. We compute the following quantities to compute the convergence rate.

For a vector field $\mathbf{w}(\mathbf{x}) = (w_1(\mathbf{x}), w_2(\mathbf{x}))$ defined on the Cartesian fluid domain $\mathbb{U}$, we define the discrete L^p norm as follows:

$$\|\mathbf{w}\|_p = \left(\sum_{\mathbf{x} \in \mathcal{G}_h} (w_1^2(\mathbf{x}) + w_2^2(\mathbf{x}))^{p/2} h^2 \right)^{1/p}, \quad 1 \le p < \infty, \tag{8.7}$$

$$\|\mathbf{w}\|_\infty = \max_{\mathbf{x} \in \mathcal{G}_h} (w_1^2(\mathbf{x}) + w_2^2(\mathbf{x}))^{1/2}. \tag{8.8}$$

Let $\mathbf{u}^N$ be the computed velocity field for the $N \times N$ mesh. We define a measure of error e_p^N as follows:

$$e_p^N = \|\mathbf{u}^N - \mathcal{I}^{2N \to N} \mathbf{u}^{2N}\|_p. \tag{8.9}$$

Here $\mathcal{I}^{2N \to N}$ is an interpolation operator from the finer to the coarser grid. As an empirical measure of global convergence rate, we use

$$r_p^N = \log_2(e_p^N / e_p^{2N}). \tag{8.10}$$

In order to examine the local convergence rate away from the immersed boundary Γ, we compute the following quantity at all points $\mathbf{x}$ on the $N \times N$ mesh that are at least two mesh widths away from the support of the discrete delta functions:

$$\rho^N(\mathbf{x}) = \log_2 \left(\frac{|\mathbf{u}^N(\mathbf{x}) - \mathbf{u}^{2N}(\mathbf{x})|}{|\mathbf{u}^{2N}(\mathbf{x}) - \mathbf{u}^{4N}(\mathbf{x})|} \right). \tag{8.11}$$

We note that this quantity can be arbitrarily large or small at certain locations since there are places where the velocity field $\mathbf{u}$ vanishes. We list the average of this quantity $\bar{\rho}^N$ as well as its mean deviation σ_ρ^N.

ϕ	$2r_\phi$	p	r_1^{256}	r_2^{256}	r_∞^{256}	$\bar{\rho}^{256}$	σ_ρ^{256}
ϕ^{ch}	2	1	1.01	1.16	0.95	0.94	0.83
ϕ^{H}	2	2	1.95	1.50	0.97	2.02	0.26
ϕ^{wH}	4	2	1.98	1.51	0.98	2.00	0.03
ϕ^{c}	4	4	1.86	1.48	0.99	2.20	0.90
ϕ^{IB}	4	2	1.98	1.50	0.98	2.00	0.01
ϕ^{IB6}	6	4	2.00	1.49	0.95	3.50	0.81

TABLE 8.1. Convergence rates for the stationary immersed boundary method for different choices of discrete delta function. $2r_\phi$ denotes the width of the support, and p the moment order. In these computations, $M = 4N$ so that $\Delta\theta$ is refined proportionally to h.

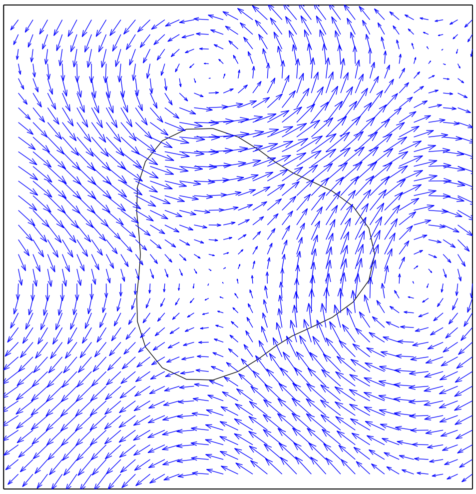

FIGURE 8.1. Velocity field computed with the immersed boundary method for the model problem. A 32×32 mesh was used to generate this figure.

We must specify $\mathbf{X}(\theta)$ and $\mathbf{F}(\theta)$. We tested various geometries and force functions. To generate Table 8.1, we used

$$\mathbf{X}(\theta) = \frac{\pi}{12}\begin{pmatrix}(6+\cos(3\theta))\cos(\theta)\\(6+\sin(3\theta))\sin(\theta)\end{pmatrix}, \quad \mathbf{F}(\theta) = \begin{pmatrix}1+\sin(\theta)\\1+\cos(\theta)\end{pmatrix}. \tag{8.12}$$

We let $M = 4N$ and $M\Delta\theta = 2\pi$ so that $\Delta\theta$ is refined proportionally to h. The velocity field as computed with this method is shown in Figure 8.1.

In Table 8.1, we list r_1^N, r_2^N, r_∞^N, $\bar{\rho}^N$, and σ_ρ^N for $N = 256$. The values at $N = 128$ were similar to those shown in the table. First of all, we see that the

ϕ	$2r_\phi$	p	r_1^{256}	r_2^{256}	r_∞^{256}	$\bar{\rho}^{256}$	σ_ρ^{256}
ϕ^{ch}	2	1	0.80	1.17	0.89	0.36	0.87
ϕ^{H}	2	2	1.95	1.50	1.00	2.03	0.26
ϕ^{wH}	4	2	1.98	1.51	0.98	2.00	0.03
ϕ^{c}	4	4	1.86	1.48	1.00	2.16	0.87
ϕ^{IB}	4	2	1.99	1.50	0.98	2.00	0.01
ϕ^{IB6}	6	4	2.00	1.49	0.95	3.50	0.81

TABLE 8.2. Convergence rates for the stationary immersed boundary method for different choices of discrete delta function. $2r_\phi$ denotes the width of the support, and p denotes the moment order. In these computations, $M = N^2/32$ so that $\Delta\theta$ is refined proportionally to h^2.

L^∞ convergence rate is always invariably close to 1. This is in agreement with the global error estimate we presented in Corollary 6.4. Except for ϕ^{ch}, whose moment order is 1, the L^1 error decays approximately at a second-order rate and the L^2 error at a rate of about 1.5.

This phenomenon can be explained as follows: We showed in Theorem 5.3 that the local convergence rate away from the boundary Γ is 2 at generic points if we refine $\Delta\theta$ proportionally to h and if the discrete delta function is of moment order at least 2. Thus, over most of the computational domain, the pointwise error must be proportional to h^2. At points within distance $\mathcal{O}(h)$ of the immersed boundary Γ, however, the error is proportional to h, as given by Corollary 6.4. Thus the L^1 error will likely behave as

$$Ch^2|\mathbb{U}| + Ch(h|\Gamma|) = \mathcal{O}(h^2) \tag{8.13}$$

where $|\mathbb{U}|$ denotes the area of the fluid domain $\mathbb{U}$ and $|\Gamma|$ the length of the curve Γ. For the L^2 norm, we have

$$(C(h^2)^2|\mathbb{U}| + Ch^2(h|\Gamma|))^{1/2} = \mathcal{O}(h^{3/2}). \tag{8.14}$$

The average local error $\bar{\rho}$ follows the trend predicted in Section 5. We see approximately second-order convergence on average for discrete delta functions of moment order 2 or higher.

In Table 8.2, we have listed the convergence rates for $M = N^2/32$ so that $\Delta\theta$ is proportional to h^2. In agreement with our theory, we do not see any significant change in the order of convergence from Table 8.1.

We make two observations that cannot be fully understood with our error analysis. The first is the behavior of ϕ^{IB6}. The local convergence rate away from the immersed boundary is approximately order 3.5 in Tables 8.1 and 8.2. For different choices **X** and **F** we have seen local convergence rates of $3.3 \sim 3.7$. Although this behavior is consistent with our error estimate, our analysis cannot explain convergence rates above order 2. This behavior cannot be due solely to the fact that ϕ^{IB6}

is of moment order 4, since ϕ^c, which is also of moment order 4, exhibits only approximately second-order local convergence in line with our predictions. We also note that ϕ^c exhibits highly erratic behavior as is evidenced by the relatively high σ_ρ-value. For different choices of **X** and **F**, $\bar{\rho}$ for ϕ^c ranges from approximately 1.5 to 2.5.

The other observation is that there are some fine details in the convergence properties that are different even among discrete delta functions of moment order 2. ϕ^{IB} and ϕ^{wH} exhibit superior convergence properties in the sense that σ_ρ is close to 0. This property was seen regardless of choice of **X** and **F**. On the other hand, the σ_ρ-value for ϕ^H is considerably higher.

A common feature of the discrete delta functions ϕ that exhibit better convergence properties is that they satisfy the following *even-odd condition* discussed briefly in Section 3:

$$\sum_{n \text{ even}} \phi(n-r) = \sum_{n \text{ odd}} \phi(n-r) \tag{8.15}$$

for all $r \in \mathbb{R}$. We believe that this condition results in cancellation of high-frequency error components in the far field.

9 Conclusion

We proved convergence of the velocity field for a simple immersed boundary method. We considered a stationary Stokes problem in a two-dimensional fluid domain with a closed one-dimensional immersed boundary curve. A key to proving convergence of the velocity field was the estimate on the difference between the discrete and continuous Green's function. We applied this estimate to prove local and global error bounds on the immersed boundary solution. The global error estimate ensures convergence of the immersed boundary solution up to the immersed boundary points.

We have not dealt with the convergence of the pressure field. The pressure field has a jump across the immersed boundary, and thus its convergence analysis may be somewhat more delicate. We hope to extend our analysis in a future paper to resolve this issue.

We used a spectral method to discretize the Stokes problem. In a forthcoming paper, we plan to generalize the results obtained in this paper to other differencing schemes.

The immersed boundary method uses a finite difference grid for the fluid domain, as was the case in this paper. There is recent work that combines some features of the immersed boundary method with finite element discretizations of the fluid domain [3, 13, 40, 41]. Such methods may be easier to study theoretically given that they are constructed within a variational framework for which powerful analytical tools are available. It would be of interest to see what bearing the present analysis may have on such methods, or whether the tools of finite element analysis may lead to a better understanding of finite-difference-based immersed boundary methods.

We hope to generalize our analysis to three-dimensional problems. Our analysis, especially near the immersed boundary points, was facilitated by the weak logarithmic singularity of the Stokeslet in two dimensions. In three dimensions the singularity of the Stokeslet is much stronger than in two dimensions.

We tested our error estimates against computational experiment. The overall features of the convergence rates are well explained by our theory. The global L^∞ convergence rate is order 1 regardless of the choice of discrete delta function as long as $\Delta\theta$ is refined proportionally to h, as indicated by our global error estimate. The local convergence rates are consistent with our theory, although some observations could not be fully understood. We believe that this is linked to the even-odd condition (8.15). A full explanation will be sought in a future study.

We applied results from the stationary problem to prove convergence of the immersed boundary solution for a small-amplitude dynamic problem. We have only considered the simplest possible elastic forcing. Typically, the elastic forcing involves derivatives in the Lagrangian coordinate. In relation to such problems, we point to [31, 34], where the authors demonstrate stability of a continuous version of such a problem for a straight filament. Such problems are yet to be studied for their discrete analogues.

One crucial aspect of the immersed boundary method that we did not touch upon is that the immersed boundary moves. We only proved convergence for a dynamic problem in which the spreading and interpolation operations were performed at fixed locations. A convergence analysis for when the boundary moves is completely open.

Convergence in the case of time-dependent Stokes or Navier-Stokes flow was not dealt with here. We have only dealt with creeping flow, in which case we could take full advantage of the instantaneous smoothing effects of the inverse Laplacian. All such questions serve as directions for future investigation.

Appendix A: Proof of Lemma 4.4

Note by assumption that neither α nor β that appears in (4.15) is equal to 1 since $0 < |x| < 2\pi$ and $0 < |y| < 2\pi$. We write $G^L_{K,ij}$ as in (4.22) and substitute expression (4.17). First, rewrite equation (4.17) as

$$
\begin{aligned}
S^l_{P,Q} &= \beta^l \frac{\alpha^P \gamma_{Pl} - \alpha^{Q+1}\gamma_{Ql}}{1-\alpha} + \sum_{k=P+1}^{Q} \frac{\alpha^k \beta^l}{1-\alpha}\epsilon_{kl}, \\
\epsilon_{kl} &= \gamma_{kl} - \gamma_{k-1,l}, \quad \gamma_{kl} = \frac{1}{|\mathbf{k}|^2}\mathcal{P}_{\mathbf{k},ij}.
\end{aligned} \tag{A.1}
$$

Substituting the above into (4.22), we find after rearrangement:

$$G^L_{K,ij} = I_L + I_K + J,$$

$$I_L = \sum_{l=-L}^{L} \frac{\beta^l \zeta_{Ll}}{1-\alpha}, \quad I_K = \sum_{l=-K}^{K} \frac{\beta^l \zeta_{Kl}}{1-\alpha}, \quad J = \sum_{k \in \tilde{\mathcal{B}}^L_K} \frac{\alpha^k \beta^l}{1-\alpha} \epsilon_{kl}, \tag{A.2}$$

$$\zeta_{Ll} = \alpha^{-L}\gamma_{-L,l} - \alpha^{L+1}\gamma_{Ll}, \quad \zeta_{Kl} = -\alpha^{-P+1}\gamma_{-P,l} + \alpha^P \gamma_{Pl},$$

where the subset $\tilde{\mathcal{B}}^L_K$ of the $\mathbf{k}$-plane is defined as

$$\tilde{\mathcal{B}}^L_K = \{\mathbf{k} \mid K < |k - \tfrac{1}{2}| < L,\ K \le |l| \le L\}. \tag{A.3}$$

To each sum in (A.2) we apply the same summation-by-parts procedure as in equation (4.16), but now in the index l. We find

$$I_L = \frac{\beta^{-L}\zeta_{-L,l} - \beta^{L+1}\zeta_{Ll}}{(1-\alpha)(1-\beta)} + \sum_{l=-L+1}^{L} \frac{\beta^l(\zeta_{Ll} - \zeta_{L,l-1})}{(1-\alpha)(1-\beta)}. \tag{A.4}$$

Thus,

$$|I_L| \le \frac{|\zeta_{-L,L}| + |\zeta_{LL}|}{|(1-\alpha)(1-\beta)|} + \sum_{l=-L+1}^{L} \frac{|\zeta_{Ll} - \zeta_{L,l-1}|}{|(1-\alpha)(1-\beta)|}. \tag{A.5}$$

First, we have

$$|\zeta_{LL}| \le |\gamma_{-L,L}| + |\gamma_{LL}| \le \frac{1}{L^2}, \tag{A.6}$$

and likewise for $\zeta_{-L,L}$. We also have

$$\begin{aligned} |\zeta_{Ll} - \zeta_{L,l-1}| &\le |\gamma_{-L,l} - \gamma_{-L,l-1}| + |\gamma_{Ll} - \gamma_{L,l-1}| \\ &\le \frac{C}{|\mathbf{k}|^3_{k=-L}} + \frac{C}{|\mathbf{k}|^3_{k=L}} \le \frac{C}{L^3}. \end{aligned} \tag{A.7}$$

We can substitute these estimates back into (A.5):

$$|I_L| \le \frac{1}{|(1-\alpha)(1-\beta)|}\left(\frac{4}{L^2} + \sum_{l=-L+1}^{L} \frac{C}{L^3}\right) \le \frac{1}{|(1-\alpha)(1-\beta)|}\frac{C}{L^2}. \tag{A.8}$$

Likewise, we have

$$|I_K| \le \frac{1}{|(1-\alpha)(1-\beta)|}\frac{C}{K^2}. \tag{A.9}$$

To estimate J in (A.2), we proceed again as in Lemma 4.3. First, define the sum

$$T^k_{P,Q} = \sum_{l=P}^{Q} \frac{\alpha^k \beta^l}{1-\alpha} \epsilon_{kl}. \tag{A.10}$$

Using the summation-by-parts argument we see that

$$|T^k_{P,Q}| = \frac{1}{|(1-\alpha)(1-\beta)|}\left(|\epsilon_{Pl}| + |\epsilon_{Ql}| + \sum_{l=P}^{Q}|\epsilon_{kl} - \epsilon_{k,l-1}|\right). \tag{A.11}$$

By direct calculation, one can check that

$$|\epsilon_{Pl}| \le \frac{C}{|\mathbf{k}|^3_{k=P}} \le \frac{C}{|P|^3}, \quad |\epsilon_{Ql}| \le \frac{C}{|Q|^3}, \quad |\epsilon_{kl} - \epsilon_{k,l-1}| \le \frac{C}{|\mathbf{k}|^4}, \tag{A.12}$$

for some constant C. Thus,

$$|T^k_{P,Q}| = \frac{C}{|(1-\alpha)(1-\beta)|}\left(\frac{1}{|P|^3} + \frac{1}{|Q|^3} + \sum_{l=P}^{Q}\frac{1}{|\mathbf{k}|^4}\right). \tag{A.13}$$

We now write J in terms of T and take absolute values:

$$\begin{aligned} |J| &\le \sum_{k=-L+1}^{-K}|T^k_{-L,L}| + \sum_{k=-K+1}^{K}(|T^l_{-L,-K}| + |T^l_{K,L}|) \\ &\quad + \sum_{l=K+1}^{L}|T^k_{-L,L}| \\ &\le \frac{C}{|(1-\alpha)(1-\beta)|}\left(\sum_{k=-L+1}^{L}\frac{2}{L^3} + \sum_{k=-K+1}^{K}\frac{2}{K^3} + \sum_{\mathbf{k}\in\mathcal{B}^L_K}\frac{1}{|\mathbf{k}|^4}\right) \\ &\le \frac{C}{|(1-\alpha)(1-\beta)|}\left(\frac{8}{K^2} + \sum_{\mathbf{k}\in\mathcal{B}^L_K}\frac{1}{|\mathbf{k}|^4}\right) \end{aligned} \tag{A.14}$$

where we used (A.13) to estimate the above sum and $L > K$ to get the last line. We now estimate the last sum:

$$\sum_{\mathbf{k}\in\mathcal{B}^L_K}\frac{1}{|\mathbf{k}|^4} \le C\int_K^{\sqrt{2}L}\frac{1}{r^4}\,r\,dr \le \frac{C}{K^2}. \tag{A.15}$$

Putting this back into (A.14), we have

$$|J| \le \frac{C}{|(1-\alpha)(1-\beta)|}\,\frac{1}{K^2}. \tag{A.16}$$

Collecting (A.8), (A.9), and (A.16) and substituting this back into (A.2), we have

$$|G^L_{K,ij}(\mathbf{x})| \le \frac{C}{K^2|(1-\alpha)(1-\beta)|}. \tag{A.17}$$

Noting that $|(1-\alpha)(1-\beta)| = 4\sin(|x|/2)\sin(|y|/2)$, we have the desired result. □

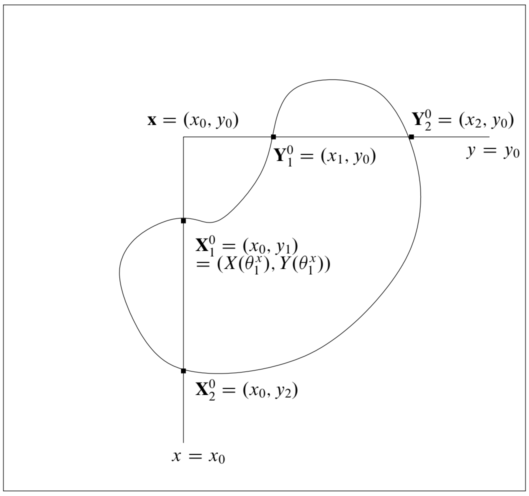

FIGURE B.1. Schematic diagram for the proof of Theorem 5.3.

Appendix B: Proof of Theorem 5.3

That nontangent points are generic is a direct consequence of Sard's theorem [22]. Given the continuous differentiable function $X(\theta)$, the values $x = X(\theta)$ at which $\partial X/\partial\theta = 0$ (critical values) form a set of measure 0 on the real line. The same is true for $y = Y(\theta)$, and thus nontangent points are generic.

We next prove that for a nontangent point $\mathbf{x_0} = (x_0, y_0)$, the coordinate lines $x = x_0$ and $y = y_0$ intersect Γ at only a finite number of points. For, assume otherwise. Suppose $x = x_0$ intersects Γ at $\mathbf{X}_k^0 = (X(\theta_k^x), Y(\theta_k^x)) = (x_0, y_k)$, $k \in \mathbb{N}$, where θ_x^k are distinct real numbers in $[0, 2\pi]$. Since the θ_k^x form a bounded sequence in $[0, 2\pi]$, there is a subsequence such that $\tilde{\theta}_k^x \to \theta^*$, $\tilde{\theta}_k^x \neq \theta^*$, for some θ^*. Then

$$\left.\frac{\partial X}{\partial \theta}\right|_{\theta=\theta^*} = \lim_{k\to\infty} \frac{x_0 - x_0}{\tilde{\theta}_k^x - \theta^*} = 0. \tag{B.1}$$

This shows that $x = x_0$ is tangent to Γ at $\mathbf{x} = \mathbf{X}(\theta^*)$ since $|\partial\mathbf{X}/\partial\theta| \neq 0$ by assumption. We thus have a contradiction. The same holds for the line $y = y_0$.

In order to prove the error estimate (5.19), we divide the difference between $\mathbf{u}$ and $\mathbf{u}_h$ as in equation (5.4). We shall derive an estimate for $\mathbf{I}^G$ that is different from the ones derived in Theorem 5.1. We shall reuse estimates (5.5) and (5.8) for $\mathbf{I}^\int$ and $\mathbf{I}^\delta$.

Let $\mathbf{X}_k^0 = (x_0, y_k)$, $k = 1, \ldots, n_x$, and $\mathbf{Y}_k^0 = (x_k, y_0)$, $k = 1, \ldots, n_y$, be the points at which the coordinate lines $x = x_0$ and $y = y_0$ intersect Γ (Figure B.1).

Let

$$\mathbf{X}_k^0 = (X(\theta_k^x), Y(\theta_k^x)) = (x_0, y_k), \quad \mathbf{Y}_k^0 = (X(\theta_k^y), Y(\theta_k^y)) = (x_k, y_0). \tag{B.2}$$

Take the arc length coordinate s along Γ. Since the coordinate lines intersect Γ at the points $\mathbf{X}_k^0$ and $\mathbf{Y}_k^0$, we have

$$\left|\frac{\partial X}{\partial s}\right|_{\mathbf{X}=\mathbf{X}_k^0} > 0, \quad \left|\frac{\partial Y}{\partial s}\right|_{\mathbf{X}=\mathbf{Y}_k^0} > 0. \tag{B.3}$$

We thus have

$$\left|\frac{\partial X}{\partial \theta}\right|_{\theta=\theta_k^x} = \left|\frac{\partial X}{\partial s}\right|_{\mathbf{X}=\mathbf{X}_k^0}\left|\frac{\partial s}{\partial \theta}\right|_{\theta=\theta_k^x} = \left|\frac{\partial X}{\partial s}\right|_{\mathbf{X}=\mathbf{X}_k^0}\left|\frac{\partial \mathbf{X}}{\partial \theta}\right|_{\theta=\theta_k^x} > 0 \tag{B.4}$$

where we used $|\partial \mathbf{X}/\partial\theta|_{\theta=\theta_k^x} \neq 0$. Likewise, we have $|\partial Y/\partial\theta|_{\theta=\theta_k^y} > 0$. Since $|\partial X/\partial\theta|_{\theta=\theta_k^x} > 0$ and $|\partial Y/\partial\theta|_{\theta=\theta_k^y}$, we can take a ρ-interval in θ around each θ_k^x and θ_k^y so that

$$\begin{aligned} \Theta_k^x &= \left\{\theta \;\middle|\; |\theta-\theta_k^x| < \rho, \left|\frac{\partial X}{\partial\theta}\right| > C_\rho > 0\right\}, \\ \Theta_k^y &= \left\{\theta \;\middle|\; |\theta-\theta_k^y| < \rho, \left|\frac{\partial Y}{\partial\theta}\right| > C_\rho > 0\right\}. \end{aligned} \tag{B.5}$$

We shall also take ρ small enough so that the intervals Θ_k^x, Θ_k^y do not overlap. Let

$$\Theta^x = \bigcup_{k=1}^{n_x} \Theta_k^x, \quad \Theta^y = \bigcup_{k=1}^{n_y} \Theta_k^y. \tag{B.6}$$

Recall $\Theta = \mathbb{R}/2\pi\mathbb{Z}$. It is clear from the definition of Θ_k^x that

$$\theta \in \Theta \setminus \Theta^x, \quad |x_0 - X(\theta)| \geq C_\rho \rho. \tag{B.7}$$

Likewise,

$$\theta \in \Theta \setminus \Theta^y, \quad |y_0 - Y(\theta)| \geq C_\rho \rho. \tag{B.8}$$

We now estimate $\mathbf{I}^G$:

$$\begin{aligned} \mathbf{I}^G &= \mathbf{J}^0 + \sum_{k=1}^{n_x} \mathbf{J}_k^x + \sum_{k=1}^{n_y} \mathbf{J}_k^y, \\ \mathbf{J}^0 &= \sum_{\theta_m \in \Theta\setminus(\Theta^x\cup\Theta^y)} (S^*(\mathbf{X})(G_{\mathbf{x}} - G_{h,\mathbf{x}}))(\theta_m)\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{J}_k^x &= \sum_{\theta_m \in \Theta_k^x} (S^*(\mathbf{X})(G_{\mathbf{x}} - G_{h,\mathbf{x}}))(\theta_m)\mathbf{F}(\theta_m)\Delta\theta, \\ \mathbf{J}_k^y &= \sum_{\theta_m \in \Theta_k^y} (S^*(\mathbf{X})(G_{\mathbf{x}} - G_{h,\mathbf{x}}))(\theta_m)\mathbf{F}(\theta_m)\Delta\theta. \end{aligned} \tag{B.9}$$

First of all, we bound $\mathbf{J}_0$. From Proposition 4.5, we have

$$
\begin{aligned}
|J_i^0| &\le \sum_{\theta_m \in \Theta \setminus (\Theta^x \cup \Theta^y)} \sum_{j=1}^{2} |(S^*(\mathbf{X})(G_{\mathbf{x},ij} - G_{h,\mathbf{x},ij}))(\theta_m)||F_j(\theta_m)|\Delta\theta \\
&\le \sum_{\theta_m \in \Theta \setminus (\Theta^x \cup \Theta^y)} \sum_{j=1}^{2} C \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta_m)}} \frac{h^2 |F_j(\theta_m)|\Delta\theta}{\sin(|x_1 - x_0|/2)\sin(|y_1 - y_0|/2)}.
\end{aligned} \tag{B.10}
$$

For $\mathbf{y} = (x_1, y_1) \in \mathcal{R}_{\mathbf{X}(\theta_m)}$, we have

$$
|x_1 - x_0| \ge |x_0 - X(\theta_m)| - |x_1 - X(\theta_m)| \ge C_\rho \rho - r_\phi h \ge C_\rho \frac{\rho}{2} \tag{B.11}
$$

where the last inequality is valid for small enough h. Likewise,

$$
|y_1 - y_0| \ge C_\rho \frac{\rho}{2} \tag{B.12}
$$

for small enough h. It is also geometrically clear that

$$
|x_1 - x_0| \le 2\pi - \frac{d_{\mathbb{U}}}{2}, \quad |y_1 - y_0| \le 2\pi - \frac{d_{\mathbb{U}}}{2}, \tag{B.13}
$$

for small enough h. This shows that $\sin(|x_1 - x_0|/2)\sin(|y_1 - y_0|/2)$ is bounded from below by a positive constant. Thus

$$
\begin{aligned}
|J_i^0| &\le \sum_{\theta_m \in \Theta \setminus (\Theta^x \cup \Theta^y)} \sum_{j=1}^{2} C \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta_m)}} \frac{h^2 |F_j(\theta_m)|\Delta\theta}{\sin(|x_1 - x_0|/2)\sin(|y_1 - y_0|/2)} \\
&\le \sum_{\theta_m \in \Theta \setminus (\Theta^x \cup \Theta^y)} \sum_{j=1}^{2} C h^2 |F_j(\theta_m)|\Delta\theta \le C h^2
\end{aligned} \tag{B.14}
$$

where the constant C above does not depend on h or $\Delta\theta$.

Now we bound $\mathbf{J}_k^x$. We first introduce the set Λ:

$$
\Lambda = \{\theta \in \Theta \mid |\theta - \theta_k^x| \le \epsilon,\ \epsilon \equiv h + r_\phi h / C_\rho\}. \tag{B.15}
$$

For h small enough, $\Lambda \subset \Theta_k^x$. We divide the sum $\mathbf{J}_k^x$ into two parts:

$$
\begin{aligned}
\mathbf{J}_k^x &= \mathbf{J}_k^{x1} + \mathbf{J}_k^{x2}, \\
\mathbf{J}_k^{x1} &= \sum_{\theta_m \in \Theta_k^x \setminus \Lambda} (S^*(\mathbf{X})(G_{\mathbf{x}} - G_{h,\mathbf{x}}))(\theta_m)\mathbf{F}(\theta_m)\Delta\theta, \\
\mathbf{J}_k^{x2} &= \sum_{\theta_m \in \Lambda} (S^*(\mathbf{X})(G_{\mathbf{x}} - G_{h,\mathbf{x}}))(\theta_m)\mathbf{F}(\theta_m)\Delta\theta.
\end{aligned} \tag{B.16}
$$

We start with $\mathbf{J}_k^{x1}$. We have

$$
|J_{k,i}^{x1}| \le \sum_{\theta_m \in \Theta_k^x \setminus \Lambda} \sum_{j=1}^{2} C \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta_m)}} \frac{h^2 |F_j(\theta_m)|\Delta\theta}{\sin(|x_1 - x_0|/2)\sin(|y_1 - y_0|/2)}. \tag{B.17}
$$

For $\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta_m)}$, $\theta_m \in \Theta_k^x \subset (\Theta \setminus \Theta^y)$,

$$2\pi - d_{\mathbb{U}} \geq |y_1 - y_0| \geq C_\rho \frac{\rho}{2}. \tag{B.18}$$

We thus see that $\sin(|y_1 - y_0|/2)$ is positive and bounded from below in the above sum. Thus,

$$|J_{k,i}^{x1}| \leq \sum_{\theta_m \in \Theta_k^x \setminus \Lambda} \sum_{j=1}^{2} C \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta_m)}} \frac{h^2 |F_j(\theta_m)| \Delta\theta}{\sin(|x_1 - x_0|/2)}. \tag{B.19}$$

Now note, for some constant C,

$$\begin{aligned} \sin\left(\frac{|x_1 - x_0|}{2}\right) &\geq C|x_1 - x_0| \geq C(|x_0 - X(\theta_m)| - r_\phi h) \\ &\geq C(C_\rho|\theta_m - \theta_k^x| - r_\phi h) \geq C(|\theta_m - \theta_k^x| - \epsilon + h) > 0. \end{aligned} \tag{B.20}$$

We thus see that

$$|J_{k,i}^{x1}| \leq \sum_{\theta_m \in \Theta_k^x \setminus \Lambda} \frac{Ch^2}{|\theta_m - \theta_k^x| - \epsilon + h} \Delta\theta. \tag{B.21}$$

We consider the part of the above sum where $\theta_m > \theta_k^x$.

$$\begin{aligned} &\sum_{\theta_m \in \Theta_k^x \setminus \Lambda, \theta_m > \theta_k^x} \frac{Ch^2 \Delta\theta}{\theta_m - \theta_k^x - \epsilon + h} \\ &\leq \frac{Ch^2}{h} \Delta\theta + Ch^2 \int_{\theta_k^x + \epsilon}^{\theta_k^x + \rho} \frac{1}{\theta - \theta_k^x - \epsilon + h} \, d\theta \\ &\leq Ch\Delta\theta + Ch^2 \int_h^\rho \frac{1}{\theta} \, d\theta \leq Ch(\Delta\theta + h\log(h^{-1})). \end{aligned} \tag{B.22}$$

The same estimate holds for the $\theta_m < \theta_k^x$ part, and we thus have

$$|J_{k,i}^{x1}| \leq Ch(\Delta\theta + h\log(h^{-1})). \tag{B.23}$$

We now turn to the estimate of $\mathbf{J}_k^{x2}$. We have

$$\begin{aligned} |J_{k,i}^{x2}| &= \sum_{\theta_m \in \Lambda} \sum_{j=1}^{2} |(S^*(\mathbf{X})(G_{\mathbf{x},ij} - G_{h,\mathbf{x},ij}))(\theta_m)| \, |F_j(\theta_m)| \Delta\theta \\ &\leq \sum_{\theta_m \in \Lambda} \sum_{j=1}^{2} C \max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\theta_m)}} \frac{h}{w(\mathbf{x} - \mathbf{y})} |F_j(\theta_m)| \Delta\theta. \end{aligned} \tag{B.24}$$

We have

$$w(\mathbf{x} - \mathbf{y}) \geq \sin\left(\frac{|y_1 - y_0|}{2}\right). \tag{B.25}$$

For $\mathbf{y} \in \Lambda$, we have (B.12), and thus $w(\mathbf{x}-\mathbf{y})$ is bounded from below by a positive constant. Thus,

$$|J_{k,i}^{x2}| \leq \sum_{\theta_m \in \Lambda} Ch\Delta\theta. \tag{B.26}$$

Since Λ is an interval of length 2ϵ, the number of θ_m contained in Λ is at most $2\epsilon/\Delta\theta + 1$. Thus,

$$|J_{k,i}^{x2}| \leq Ch\Delta\theta\left(2\frac{\epsilon}{\Delta}\theta + 1\right) \leq Ch(h + \Delta\theta). \tag{B.27}$$

From (B.23) and (B.27) we have

$$|\mathbf{J}_k^x| \leq Ch(\Delta\theta + h\log(h^{-1})). \tag{B.28}$$

The same estimate holds for $\mathbf{J}_k^y$. Substituting this and (B.14) into (B.9) we have

$$\begin{aligned}
|\mathbf{I}^G| &\leq |\mathbf{J}^0| + \sum_{k=1}^{n_x}|\mathbf{J}_k^x| + \sum_{k=1}^{n_y}|\mathbf{J}_k^y| \\
&\leq Ch^2 + \sum_{k=1}^{n_x} Ch(\Delta\theta + h\log(h^{-1})) + \sum_{k=1}^{n_y} Ch(\Delta\theta + h\log(h^{-1})) \\
&\leq Ch(\Delta\theta + h\log(h^{-1})).
\end{aligned} \tag{B.29}$$

Combining this with (5.5) and (5.8) from the proof of Theorem 5.1, we have the desired result. □

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