L^P CONVERGENCE OF THE IMMERSED BOUNDARY METHOD FOR STATIONARY STOKES PROBLEMS

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Abstract. In this paper, we analyze the convergence of the immersed boundary (IB) method as applied to a static Stokes flow problem. Using estimates obtained in [5], we consider a problem in which a d-dimensional structure is immersed in n-dimension, and prove error estimates for both the pressure and the velocity field in the $L^p(1 \leq p \leq \infty)$ norm. One interesting consequence of our analysis is that the asymptotic error rates in the L^1 norm do not depend on either d or n and in the L^p $(p > 1)$ norm they only depend on $n - d$. The resulting estimates are checked numerically for optimality.

Key words. immersed boundary method, L^p error estimates, discrete delta functions.

1. Introduction. The immersed boundary method has been widely used to solve problems with moving interfaces (fluid-structure interaction, two phase fluid flow, etc) along which variables of interest often possess discontinuities. In IB formulations, these problems are recast as partial differential equations (PDE) over a simpler domain with singular source terms distributed along the interfaces. Dirac delta functions are used to represent the singular source term as a distribution defined on the entire fluid domain. The PDE are often discretized on a Eulerian grid over the fluid domain, while the singular source term is often discretized by integrating over a Lagrangian grid on the interface using regularized Dirac delta functions (often called discrete delta functions) . The discretized equations are then often solved with standard methods.

In addition to the IB method, many other methods are often used to solve such free surface problems, with boundary integral methods and level set methods being prime examples. Among these methods, algorithms based on the IB method are often very efficient and easy to implement. Although it has been justified by numerical results in practice, convergence of the IB method is often unresolved from an analytical point of view. Despite being the topic of many papers [6, 8, 9, 1, 3, 4], convergence analysis of IB methods is still at a primary stage. As a first step to analyze full dynamic problems, convergence properties of the IB method were studied for a stationary Stokes flow problem in [6]. For the velocity field, point-wise and L^∞ error estimates are obtained. Then as an application of the results, L^2 error estimates are studied for a simple dynamic problem. The analysis relies on the geometric structure of the 2D model problem, which is a one-dimensional elastic string immersed in a two-dimensional fluid domain.

In [5], the analysis is extended to a more general elliptic model problem. It is worth mentioning that the analysis in [5], despite carried through for an immersed boundary method setup, is developed for more general problems involving discontinuities that are regularized by using discrete delta functions. For an immersed boundary model problem with general choices of n and d, where n is the dimension of the fluid domain and d is the dimension of the immersed structure, the error is decomposed and carefully studied. Some estimates are established for the part of the error that is related to the immersed boundary method and essentially the discrete delta function in use. It is shown that the convergence properties depend on certain properties of

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the discrete delta function, the involved differential operators, accuracy of the discretization schemes used to discretize the spatial derivatives, and regularities of the interface and the forces exerted along it. As an application, the results are used to analyze point-wise convergence behaviors of the velocity field for a 2D Stokes flow immersed boundary problem.

The goal of this paper is to explain how to obtain L^p ($1 \leq p \leq \infty$) error estimates for similar problems. We choose a Stokes flow immersed boundary model problem which is similar to the 2D model problem in [5] but with general choices of n and d, where $1 \leq d \leq n$. The analysis is based on the point-wise error estimates and essentially the immersed boundary error estimates (Theorem 4.6) in [5]. In this sense, this paper can be seen as a sequel to the latter. We establish L^p error estimates for not only the velocity but also the pressure. It is interesting to see that the L^1 error estimates basically do not depend on either n or d. For all $1 < p \leq \infty$, the L^p error estimates only depend on $n-d$. To focus on the impact of the properties of the discrete delta function on the error estimates, we use spectral schemes for the spatial derivatives in the model problem. But the analysis is written in a general framework that requires little modification if instead conventional finite difference schemes are used. It can be seen in the proofs of the theorems how the convergence properties are determined by all these factors. For example, the smoothing order of the discrete delta function, introduced in [5], only affects the asymptotic error rates via the relatively negligible logarithmic terms while it is the moment order, the accuracy of the spatial differentiation schemes and the regularity of the involved functions that can be dominant factors.

We now give a brief outline of this paper. In Section 2, we introduce the model problem, formulate and decompose the errors. In Section 3, we provide some technical results that will be used in developing the L^p error estimates. In Section 4, the L^1 error estimates are established. Section 5 discusses the L^p error estimates for all $1 \leq p \leq \infty$, when $n \geq 2, n-1 \leq d \leq n$. In Section 6, we test the predicted results in numerical experiments for optimality. Section 7 gives a summary of the results and includes a short discussion on possible relaxations of the assumptions. Appendix A contains some estimates for the Green's functions that are needed in the convergence analysis.

2. Model Problem and Error Decomposition. In this section we state the model problem and decompose the errors to facilitate further analysis.

2.1. Model Problem. Consider a Stokes flow problem on an n-dimensional ($n \geq 2$) periodic fluid domain $\mathbb{U} = (\mathbb{R}/2\pi\mathbb{Z})^n \subset \mathbb{R}^n$ with a d-dimensional immersed structure $\Gamma \subset \mathbb{U}$, where $1 \leq d \leq n$. The immersed structure Γ is parameterized by a vector-valued function $\mathbf{X}(\boldsymbol{\theta}) = (X_1(\boldsymbol{\theta}), X_2(\boldsymbol{\theta}), \cdots, X_n(\boldsymbol{\theta}))$, where $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_d)$ on a set $\Theta \subseteq \mathbb{R}^d$. In other words, $\Gamma = \mathbf{X}(\Theta)$. For simplicity, when $d < n$ we choose Θ to be a square $[0, 2\pi]^d$ and when $d = n$ we choose Θ to be a torus, that is, $\Theta = (\mathbb{R}/2\pi\mathbb{Z})^d$. We suppose that Γ is away from the boundary of $\mathbb{U}$, i.e., $\Gamma \subset\subset \mathbb{U}$. Let $\mathbf{F}(\boldsymbol{\theta})$ denote the force distributed along Γ. Throughout this paper, we assume that $\mathbf{X}$ is a C^2-diffeomorphism between Θ and Γ and $\mathbf{F}$ is also a C^2 function on Θ. In the IB method formulation we spread $\mathbf{F}$ over the entire domain $\mathbb{U}$ using Dirac delta functions and represent the problem as the following equations:

$$\Delta \mathbf{u} = \nabla P - \mathbf{f} + \mathbf{g}, \quad \nabla \cdot \mathbf{u} = 0, \tag{2.1}$$

$$\mathbf{f}(\mathbf{x}) = \int_\Theta \mathbf{F}(\boldsymbol{\theta})\delta(\mathbf{x} - \mathbf{X}(\boldsymbol{\theta}))d\boldsymbol{\theta}, \tag{2.2}$$

$$\mathbf{g} = \frac{1}{(2\pi)^n}\int_\Theta \mathbf{F}(\boldsymbol{\theta})d\boldsymbol{\theta}, \quad \int_{\mathbb{U}} \mathbf{u}\, d\mathbf{x} = 0, \quad \int_{\mathbb{U}} P\, d\mathbf{x} = 0, \tag{2.3}$$

where all the equations are rendered dimensionless with the dynamic viscosity $\mu = 1$, $\mathbf{u}$ is the velocity vector, P is the pressure, $\mathbf{f}$ is the source term generated by spreading $\mathbf{F}$ over $\mathbb{U}$ and δ is the Dirac delta function. As in [5, 6], the equations in (2.3) are imposed to ensure unique solutions.

The problem is discretized as follows. We first lay a uniform Eulerian grid $\mathcal{G}_h$ with grid width h on $\mathbb{U}$ and a uniform Lagrangian grid $\mathcal{G}_\theta$ with grid width $\Delta\theta$ on Γ. In this paper $\Delta\theta$ and h are assumed to be proportional to each other to avoid nonessential complexities. We place the grids in the way that coordinates of the grid points on $\mathcal{G}_h$ are all multiples of h and coordinates of the grid points on $\mathcal{G}_\theta$ are all of the form $\frac{\Delta\theta}{2} + i\Delta\theta$, for some $i = 0, 1, \cdots, \frac{2\pi}{\Delta\theta} - 1$, that is, the Lagrangian grid starts $\Delta\theta/2$ away from the edges of Θ. A second order mid-point rule is used to discretize the integrals on Θ. The equations (2.1), (2.2) and (2.3) are discretized as follows.

$$\Delta_h \mathbf{u}_h = \nabla_h P_h - \mathbf{f}_h + \mathbf{g}_h, \quad \nabla_h \cdot \mathbf{u}_h = 0, \tag{2.4}$$

$$\mathbf{f}_h = \sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} \mathbf{F}(\hat{\boldsymbol{\theta}})\delta_h(\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}}))(\Delta\theta)^d, \tag{2.5}$$

$$\mathbf{g}_h = \frac{1}{(2\pi)^n}\sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} \mathbf{F}(\hat{\boldsymbol{\theta}})(\Delta\theta)^d, \quad \sum_{\mathbf{x}\in\mathcal{G}_h} \mathbf{u}_h(\mathbf{x})h^n = 0, \quad \sum_{\mathbf{x}\in\mathcal{G}_h} P_h(\mathbf{x})h^n = 0, \tag{2.6}$$

where Δ_h and ∇_h are spectral or finite difference discretizations of Δ and ∇ respectively and δ_h is the discrete delta function used to regularize the Dirac delta function δ. As explained in [5], usually we should choose schemes of order $q \geq 2$ for Δ_h and ∇_h, which basically means the consistency errors from using Δ_h and ∇_h to discretize Δ and ∇ are $O(h^q)$. The precise meaning of q is defined by e.q.(4.15) in [5]. In this remaining part of this paper, we use spectral schemes for Δ_h and ∇_h in which case $q = \infty$. Therefore the accuracy of the schemes does not affect the convergence behaviors, and we can focus on the other factors in which we are mainly interested. We point out that the analysis also works for conventional finite difference schemes since it is based on the results in [5] which are developed for general discretization schemes of order q. Assume the discrete delta function δ_h has the following form:

$$\delta_h(\mathbf{x}) = \frac{1}{h^n}\prod_{i=1}^{n} \phi\left(\frac{x_i}{h}\right), \ \mathbf{x} = (x_1, \cdots, x_n)^T, \tag{2.7}$$

for some function ϕ defined on $\mathbb{R}$. Following [5], we impose the following conditions on ϕ:

- ϕ is compactly supported.
- ϕ is of *moment order* m ($m \geq 1$), that is,

$$\sum_{k\in\mathbb{Z}} \phi(k - r) = 1, \tag{2.8}$$

$$\sum_{k\in\mathbb{Z}} (k - r)^j \phi(k - r) = 0 \quad \text{for all } 1 \leq j \leq m - 1, \tag{2.9}$$

for all $r \in \mathbb{R}$. In this paper, we always assume $m \geq 2$ which is usually necessary for obtaining enough accuracy or even convergence in practice.

- ϕ is of *smoothing order* s ($s \geq 0$), which is defined as the following: If $s \geq 1$, then there is a function $\psi(r)$ of compact support such that

$$\phi(r) = \frac{1}{2^s} \sum_{l=0}^{s} \binom{s}{l} \psi(r-l), \tag{2.10}$$

where $\binom{s}{l}$ is the binomial coefficient; if $s = 0$, then it just means that ϕ is compactly supported.

The condition for compact support is imposed for computational efficiency. We denote by a_ϕ the smallest positive half integer (i.e., $2a_\phi$ is an integer) such that $\phi(r) \neq 0$ only if $-a_\phi \leq r < a_\phi$. For any $X \in \mathbb{U}$, let $\mathcal{R}_{\mathbf{X}} = \{\mathbf{x} : |x_i - X_i| \leq a_\phi h, i = 1, \cdots, n\}$. As suggested in [8, 6, 5], the moment order controls the accuracy of the interpolation operation (see equation (2.22)). The smoothing order was introduced in [2] and [5], and has the effect of taming error components with high spatial frequency. In addition, sometimes ϕ is assumed to satisfy the following condition [7]:

$$\sum_j (\phi(r-j))^2 = C, \quad \text{for all } r \in \mathbb{R}, \tag{2.11}$$

for some constant C independent of r. In this paper we do not consider this condition.

Our convergence analysis relies on proper representations of the errors $\mathbf{u} - \mathbf{u}_h$ and $P - P_h$. In [5], we divide the error into two parts, the *quadrature error* and the *immersed boundary error* and have established estimates (Theorem 4.6) for the latter. We decompose the errors similarly in the following section.

2.2. Error Decomposition. Following [6] and [5], we represent $\mathbf{u}$, P, $\mathbf{u}_h$ and P_h using Green's functions. We use G and Π to denote the continuous Green's functions for the velocity field and the pressure respectively. Similarly G_h and Π_h are used for the corresponding discrete Green's functions. We write G, Π, G_h and Π_h as Fourier sums:

$$\mathbf{G}(\mathbf{x}) = \frac{1}{(2\pi)^n} \sum_{|\mathbf{k}| \neq 0} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{1}{|\mathbf{k}|^2} \mathcal{P}_{\mathbf{k}}, \tag{2.12}$$

$$\mathbf{\Pi}(\mathbf{x}) = \frac{1}{(2\pi)^n} \sum_{|\mathbf{k}| \neq 0} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{1}{|\mathbf{k}|} \mathcal{Q}_{\mathbf{k}}, \tag{2.13}$$

$$\mathbf{G}_h(\mathbf{x}) = \frac{1}{(2\pi)^n} \sum_{\mathbf{k} \in \mathcal{K}_h} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{1}{|\mathbf{k}|^2} \mathcal{P}_{\mathbf{k}}, \tag{2.14}$$

$$\mathbf{\Pi}_h(\mathbf{x}) = \frac{1}{(2\pi)^n} \sum_{\mathbf{k} \in \mathcal{K}_h} \exp(i\mathbf{k}\cdot\mathbf{x}) \frac{1}{|\mathbf{k}|} \mathcal{Q}_{\mathbf{k}}, \tag{2.15}$$

$$\mathcal{K}_h = \{\mathbf{k} \in \mathbb{R}^n : -\pi \leq k_i h < \pi, \ i = 1, 2, \cdots, n\}, \tag{2.16}$$

$$\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \quad \mathcal{P}_{\mathbf{k}} = \frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2} - I_n, \quad \mathcal{Q}_{\mathbf{k}} = \frac{i}{|\mathbf{k}|} \begin{pmatrix} k_1 & \cdots & k_n \end{pmatrix}, \tag{2.17}$$

where I_n is the $n \times n$ identity matrix, and the components $k_1, k_2, \cdots, k_n$ of the vector $\mathbf{k}$ are integers. We comment that if instead here some conventional finite difference schemes are used for Δ_h and ∇_h, then G_h and Π_h can be written in the same form and the analysis can be carried out in a similar fashion. The coefficients in the Fourier sums will be determined by the Fourier symbols of the discretized operators and behave similarly to $\mathcal{P}_{\mathbf{k}}$ and $\mathcal{Q}_{\mathbf{k}}$. Here we only provide details for the case where spectral schemes are used. Write $\mathbf{u}$, P, $\mathbf{u}_h$ and P_h as follows:

$$\mathbf{u}(\mathbf{x}) = \int_\Theta \mathbf{G}(\mathbf{x} - \mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})d\boldsymbol{\theta}, \tag{2.18}$$

$$P(\mathbf{x}) = \int_\Theta \mathbf{\Pi}(\mathbf{x} - \mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})d\boldsymbol{\theta}, \tag{2.19}$$

$$\mathbf{u}_h(\mathbf{x}) = \sum_{\boldsymbol{\theta}\in\mathcal{G}_\theta} (\mathcal{I}\mathbf{G}_{h,\boldsymbol{x}})(\boldsymbol{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})(\Delta\theta)^d, \tag{2.20}$$

$$P_h(\mathbf{x}) = \sum_{\boldsymbol{\theta}\in\mathcal{G}_\theta} (\mathcal{I}\mathbf{\Pi}_{h,\boldsymbol{x}})(\boldsymbol{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})(\Delta\theta)^d, \tag{2.21}$$

where $\mathbf{G}_{h,\boldsymbol{x}}(\mathbf{y}) = \mathbf{G}_h(\mathbf{x} - \mathbf{y})$ and $\mathbf{\Pi}_{h,\boldsymbol{x}}(\mathbf{y}) = \mathbf{\Pi}_h(\mathbf{x} - \mathbf{y})$. For a function q defined on the grid $\mathcal{G}_h$, define

$$(\mathcal{I}q)(\mathbf{Y}) = \sum_{\mathbf{y}\in\mathcal{G}_h} q(\mathbf{y})\delta_h(\mathbf{y} - \mathbf{Y})h^n, \tag{2.22}$$

where $\mathbf{Y} \in \mathbb{U}$. The function $\mathcal{I}q$ defined on $\mathbb{U}$ can be seen as an interpolant of the grid function q and for this reason the operator $\mathcal{I}$ is called the *interpolation operator*. In this model problem $\mathcal{I}$ may act on vectors and matrices, with the understanding that it acts on each component separately. The goal is to obtain L^p $(1 \le P \le \infty)$ estimates of the following errors:

$$\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x}) = \int_\Theta \mathbf{G}(\mathbf{x} - \mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})d\boldsymbol{\theta} - \sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} (\mathcal{I}\mathbf{G}_{h,\boldsymbol{x}})(\boldsymbol{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}})(\Delta\theta)^d, \tag{2.23}$$

$$P(\mathbf{x}) - P_h(\mathbf{x}) = \int_\Theta \mathbf{\Pi}(\mathbf{x} - \mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})d\boldsymbol{\theta} - \sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} (\mathcal{I}\mathbf{\Pi}_{h,\boldsymbol{x}})(\boldsymbol{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}})(\Delta\theta)^d. \tag{2.24}$$

In [5] we split the error into two parts: the *quadrature error* and the *immersed boundary error*, namely the error from discrete integration and the error from using the IB method. For example, the error for the velocity field is divided as follows:

$$\begin{aligned} &\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x}) = \mathbf{E}_Q^{ve}(\mathbf{x}) + \mathbf{E}_{IB}^{ve}(\mathbf{x}), \\ &\quad \mathbf{E}_Q^{ve}(\mathbf{x}) = \int_\Theta \mathbf{G}_{\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})d\theta - \sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} \mathbf{G}_{\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}})(\Delta\theta)^d, \\ &\quad \mathbf{E}_{IB}^{ve}(\mathbf{x}) = \sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} \boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}), \mathbf{x})\mathbf{F}(\hat{\boldsymbol{\theta}})(\Delta\theta)^d, \quad \boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}, \mathbf{x}) = (\mathbf{G}_{\mathbf{x}} - \mathcal{I}\mathbf{G}_{h,\mathbf{x}})(\mathbf{X}), \end{aligned} \tag{2.25}$$

where $\mathbf{E}_Q^{ve}(\mathbf{x})$ is the quadrature error and $\mathbf{E}_{IB}^{ve}(\mathbf{x})$ is the immersed boundary error. The estimates of the quadrature errors from using a mid-point rule depend on $\mathbf{G}$ and its derivatives. Some useful estimates of these Green's functions and their derivatives when $n = 2$ are provided in Appendix A and for higher dimensional problems the results are similar. The result (Theorem 4.6) obtained in [5] provides some comprehensive estimates for $\boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}, \mathbf{x})$.

Here we take a similar approach. In order to obtain the L^p error estimates, we first divide Θ. For any $\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_d) \in \mathcal{G}_\theta$, let

$$\mathbf{I}_{\hat{\boldsymbol{\theta}}} = [\hat{\theta}_1 - \frac{\Delta\theta}{2}, \hat{\theta}_1 + \frac{\Delta\theta}{2}) \times [\hat{\theta}_2 - \frac{\Delta\theta}{2}, \hat{\theta}_2 + \frac{\Delta\theta}{2}) \times \cdots \times [\hat{\theta}_d - \frac{\Delta\theta}{2}, \hat{\theta}_d + \frac{\Delta\theta}{2}), \quad (2.26)$$

then $\Theta = \cup_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} \mathbf{I}_{\hat{\boldsymbol{\theta}}}$. Write $\mathbf{u} - \mathbf{u}_h$ and $P - P_h$ as follows:

$$\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x}) = \sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} \mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})(\Delta\theta)^d, \quad (2.27)$$

$$P(\mathbf{x}) - P_h(\mathbf{x}) = \sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} N(\hat{\boldsymbol{\theta}}, \mathbf{x})(\Delta\theta)^d, \quad (2.28)$$

where

$$\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x}) = \frac{1}{(\Delta\theta)^d} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} \mathbf{G}_{\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta}) d\boldsymbol{\theta} - \mathcal{I}\mathbf{G}_{h,\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}}), \quad (2.29)$$

$$N(\hat{\boldsymbol{\theta}}, \mathbf{x}) = \frac{1}{(\Delta\theta)^d} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} \boldsymbol{\Pi}_{\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta}) d\boldsymbol{\theta} - \mathcal{I}\boldsymbol{\Pi}_{h,\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}}). \quad (2.30)$$

Furthermore, define

$$\boldsymbol{\mathcal{E}}_Q^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}), \mathbf{x}) = \frac{1}{(\Delta\theta)^d} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} \mathbf{G}_{\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta}) d\boldsymbol{\theta} - \mathbf{G}_{\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}}), \quad (2.31)$$

$$\mathcal{E}_Q^{pr}(\mathbf{X}(\hat{\boldsymbol{\theta}}), \mathbf{x}) = \frac{1}{(\Delta\theta)^d} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} \boldsymbol{\Pi}_{\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta}) d\boldsymbol{\theta} - \boldsymbol{\Pi}_{\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}}). \quad (2.32)$$

Then

$$\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x}) = \boldsymbol{\mathcal{E}}_Q^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}), \mathbf{x}) + \boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}), \mathbf{x})\mathbf{F}(\hat{\boldsymbol{\theta}}), \quad (2.33)$$

$$N(\hat{\boldsymbol{\theta}}, \mathbf{x}) = \mathcal{E}_Q^{pr}(\mathbf{X}(\hat{\boldsymbol{\theta}}), \mathbf{x}) + \mathcal{E}_{IB}^{pr}(\mathbf{X}(\hat{\boldsymbol{\theta}}), \mathbf{x})\mathbf{F}(\hat{\boldsymbol{\theta}}), \quad (2.34)$$

where $\boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}, \mathbf{x}) = (\mathbf{G}_{\mathbf{x}} - \mathcal{I}\mathbf{G}_{h,\mathbf{x}})(\mathbf{X})$, $\boldsymbol{\mathcal{E}}_{IB}^{pr}(\mathbf{X}, \mathbf{x}) = (\boldsymbol{\Pi}_{\mathbf{x}} - \mathcal{I}\boldsymbol{\Pi}_{h,\mathbf{x}})(\mathbf{X})$. We rely on (2.27) and (2.28) to develop the L^p error estimates. The needed estimates of $\mathbf{M}$ and N will be discussed in the next section.

At the end of this section, we state a result from [6] on the boundedness of the interpolation operator $\mathcal{I}$ defined in (2.22) that will be used later in developing the L^p error estimates.

Lemma 2.1. *Let $q(\mathbf{x})$ be a function defined on the fluid grid $\mathcal{G}_h$. When interpolating at a point $\mathbf{X}_0 = \mathbf{X}(\boldsymbol{\theta}_0)$, we have*

$$|(\mathcal{I}q)\mathbf{X}_0| \leq C \max_{\mathbf{x} \in \mathcal{R}_{\mathbf{X}_0}} |q(\mathbf{x})|, \quad (2.35)$$

for some constant $C > 0$ that depends only on ϕ.

3. Estimates of M and N. Consider two points $\mathbf{x} \in \mathbb{U}$ and $\mathbf{X}(\hat{\boldsymbol{\theta}})$, $\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta$. As mentioned earlier, our approach to obtaining the L^p error estimates relies on the estimates of $\mathbf{M}$ and N, which are largely based on Theorem 4.6 in [5] and hence depend on $\mathbf{x} - \mathbf{X}(\boldsymbol{\theta})$, $\boldsymbol{\theta} \in \mathbf{I}_{\hat{\boldsymbol{\theta}}}$. Recall that by assumption $\mathbf{X}$ is a C^2 diffeomorphism between Θ and Γ.

LEMMA 3.1. *For the model problem* ($n \geq 2$, $1 \leq d \leq n$), *there exists a uniform constant* $\rho > 0$ *such that for any* $\boldsymbol{\theta}_0 \in \Theta$, *we can find* d *mutually different integers* $l_1, l_2, \cdots, l_d \in \{1, 2, \cdots, n\}$ *such that*

$$\left| \frac{\partial X_{l_i}(\boldsymbol{\theta}_0)}{\partial \theta_i} \right| \geq \rho, \tag{3.1}$$

for all $i = 1, 2, \cdots, d$. *As a result, for any* $\boldsymbol{\theta} \in \Theta$, $\boldsymbol{\gamma} \in \Theta$, *when* $\boldsymbol{\theta}$ *and* $\boldsymbol{\gamma}$ *are sufficiently close or when* $\mathbf{X}(\boldsymbol{\theta})$ *and* $\mathbf{X}(\boldsymbol{\gamma})$ *are sufficiently close, the distances* $|\boldsymbol{\theta} - \boldsymbol{\gamma}|$ *and* $|\mathbf{X}(\boldsymbol{\theta}) - \mathbf{X}(\boldsymbol{\gamma})|$ *are comparable. In precise, there exists a sufficiently small constant* $\epsilon > 0$ *and constants* C, C', $0 < C < C'$, *all of which only depend on the geometry of* $\mathbf{X}$ *such that for any* $\boldsymbol{\theta} \in \Theta$, $\boldsymbol{\gamma} \in \Theta$, *if* $|\boldsymbol{\theta} - \boldsymbol{\gamma}| \leq \epsilon$ *or* $|\mathbf{X}(\boldsymbol{\theta}) - \mathbf{X}(\boldsymbol{\gamma})| \leq \epsilon$, *we have*

$$C\,|\boldsymbol{\theta} - \boldsymbol{\gamma}| \leq |\mathbf{X}(\boldsymbol{\theta}) - \mathbf{X}(\boldsymbol{\gamma})| \leq C'\,|\boldsymbol{\theta} - \boldsymbol{\gamma}|\,. \tag{3.2}$$

Proof. By assumption $\mathbf{X}$ is nonsingular on Θ, that is, the following function Λ is positive at any point $\boldsymbol{\theta}_0 \in \Theta$:

$$\Lambda(\boldsymbol{\theta}_0) = \sum_{\substack{1,\cdots,l_d \in \{1,2,\cdots,n\} \\ l_i \neq l_j \text{ if } i \neq j}} \left| \det \left| \frac{\partial\left(X_{l_1}, X_{l_2}, \cdots, X_{l_d}\right)}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta} = \boldsymbol{\theta}_0} \right|. \tag{3.3}$$

Since Λ is continuous and defined on a compact set , there exists a constant $C_1 > 0$ such that for any $\boldsymbol{\theta}_0 \in \Theta$,

$$\Lambda(\boldsymbol{\theta}_0) \geq C_1. \tag{3.4}$$

Then from the definition of Λ, we know there exists a certain choice of $l_1, \cdots, l_d$ such that

$$\left| \det \left| \frac{\partial\left(X_{l_1}, X_{l_2}, \cdots, X_{l_d}\right)}{\partial \boldsymbol{\theta}} \right|_{\boldsymbol{\theta} = \boldsymbol{\theta}_0} \right| \geq C_2 \tag{3.5}$$

where $C_2 > 0$ is a constant determined by C_1. By assumption $\mathbf{X}$ is C^1 on a compact set and hence $\left| \frac{\partial X_{l_i}(\boldsymbol{\theta}_0)}{\partial \theta_j} \right|$ is uniformly bounded for any $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, d$. Expanding (3.5) and choosing a proper ρ, we get (3.1). The result (3.2) is an immediate conclusion when ϵ if sufficiently small. ∎

From Lemma 3.1 and the assumption on h and $\Delta\theta$ being proportional to each other, we know that when h and $\Delta\theta$ are sufficiently small, for any $\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta$, $\mathbf{Y} \in \mathcal{R}_{\mathbf{X}(\hat{\boldsymbol{\theta}})} \cup \mathbf{X}(\mathbf{I}_{\hat{\boldsymbol{\theta}}})$, we have $\left|\mathbf{Y} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right| \leq \frac{\lambda}{2}\Delta\theta$, for some constant $\lambda > 0$ that does not depend on $\hat{\boldsymbol{\theta}}$ or $\mathbf{Y}$. Then as an immediate result, for all $\mathbf{x} \in \mathbb{U}$ such that $\left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right| > \lambda\Delta\theta$, we have

$$\frac{1}{2}\left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right| < |\mathbf{x} - \mathbf{Y}| < \frac{3}{2}\left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right|, \tag{3.6}$$

for all $\mathbf{Y} \in \mathcal{R}_{\mathbf{X}(\hat{\boldsymbol{\theta}})} \cup \mathbf{X}(\mathbf{I}_{\hat{\boldsymbol{\theta}}})$. In this case, the estimates of $\mathbf{M}$ and N only depend on $\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})$ and are stated as follows.

LEMMA 3.2. *When* $\left|\mathbf{X}(\hat{\boldsymbol{\theta}})-\mathbf{x}\right| \geq \lambda\Delta\theta$, *we have*

$$\left|M_i(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C\Delta\theta}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|}, \tag{3.7}$$

$$\left|M_i(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C(\Delta\theta)^2\log(\Delta\theta)^{-1}}{\left|x_1-X_1(\hat{\boldsymbol{\theta}})\right|\cdot\left|x_2-X_2(\hat{\boldsymbol{\theta}})\right|}, \tag{3.8}$$

$$\left|M_i(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C(\Delta\theta)^2}{\left|x_1-X_1(\hat{\boldsymbol{\theta}})\right|\cdot\left|x_2-X_2(\hat{\boldsymbol{\theta}})\right|} \quad \textit{if} \quad m\geq 3, \tag{3.9}$$

$$\left|M_i(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C(\Delta\theta)^2(\log\Delta\theta)^{-1}}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^2} \quad \textit{if} \quad s\geq 1, \tag{3.10}$$

$$\left|M_i(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C(\Delta\theta)^2}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^2} \quad \textit{if} \quad m\geq 3,\ s\geq 1, \tag{3.11}$$

$$\left|N(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C\log(\Delta\theta)^{-1}}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|}, \tag{3.12}$$

$$\left|N(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C\Delta\theta}{\left|x_1-X_1(\hat{\boldsymbol{\theta}})\right|\cdot\left|x_2-X_2(\hat{\boldsymbol{\theta}})\right|}, \tag{3.13}$$

$$\left|N(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C\Delta\theta}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^2} \quad \textit{if}\, s\geq 1, \tag{3.14}$$

$$\left|N(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C(\Delta\theta)^2\log(\Delta\theta)^{-1}}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^3} \quad \textit{if}\, s\geq 2, \tag{3.15}$$

$$\left|N(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \leq \frac{C(\Delta\theta)^2}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^3}, \quad \textit{if} \quad m\geq 3,\ s\geq 2,, \tag{3.16}$$

where $i=1,2$ *and* $C>0$ *denote uniform constants that do not depend on* $\mathbf{x}$, $\hat{\boldsymbol{\theta}}$ *or* h.

Proof. First divide $\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})$ as in (2.33). The mid-point rule is of second order accuracy and hence from the assumption that $\mathbf{X}$ is C^2 we have

$$\begin{aligned}\left|\boldsymbol{\mathcal{E}}_Q^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}),\mathbf{x})\right| &\leq C\sum_{j=1}^{2}\max_{\substack{\theta=(\theta_1,\theta_2)\in\mathbf{I}_{\hat{\boldsymbol{\theta}}}\\ \alpha_1+\alpha_2=2}}\left|\frac{\partial^{\alpha_1+\alpha_2}G_{ij,\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))}{\partial\theta_1^{\alpha_1}\partial\theta_2^{\alpha_2}}\right|(\Delta\theta)^2\\ &\leq C\frac{1}{\min\limits_{\boldsymbol{\theta}\in\mathbf{I}_{\hat{\boldsymbol{\theta}}}}\left|\mathbf{x}-\mathbf{X}(\boldsymbol{\theta})\right|^2}\cdot(\Delta\theta)^2 \leq \frac{C}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^2}\cdot(\Delta\theta)^2,\end{aligned} \tag{3.17}$$

where $\alpha_i = 0,1,2,\ i=1,2$ and $C>0$ denote uniform constants. In the second inequality we used Lemma A.1 and in the third inequality we used (3.6) from the definition of λ. Similarly, only using the assumption that $\mathbf{X}$ is C^1, we have

$$\left|\boldsymbol{\mathcal{E}}_Q^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}),\mathbf{x})\right| \le C\frac{\Delta\theta}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|}. \tag{3.18}$$

On the other hand, by Theorem 4.6 in [5], we have

$$\left|\boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}),\mathbf{x})\right| \le \frac{Ch}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|}, \tag{3.19}$$

$$\left|\boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}),\mathbf{x})\right| \le \frac{Ch^2\log h^{-1}}{\left|x_1-X_1(\hat{\boldsymbol{\theta}})\right|\cdot\left|x_2-X_2(\hat{\boldsymbol{\theta}})\right|}, \tag{3.20}$$

$$\left|\boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}),\mathbf{x})\right| \le \frac{Ch^2}{\left|x_1-X_1(\hat{\boldsymbol{\theta}})\right|\cdot\left|x_2-X_2(\hat{\boldsymbol{\theta}})\right|} \quad \text{if} \quad m\ge 3, \tag{3.21}$$

$$\left|\boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}),\mathbf{x})\right| \le \frac{Ch^2\log h^{-1}}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^2} \quad \text{if} \quad s\ge 1, \tag{3.22}$$

$$\left|\boldsymbol{\mathcal{E}}_{IB}^{ve}(\mathbf{X}(\hat{\boldsymbol{\theta}}),\mathbf{x})\right| \le \frac{Ch^2}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|^2} \quad \text{if} \quad m\ge 3,\ s\ge 1. \tag{3.23}$$

Collecting these results and using the assumption that $\mathbf{F}$ is C^2 on a compact set, we obtain the estimates (3.7), (3.8), (3.9), (3.10) and (3.11) for $\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})$. For $N(\hat{\boldsymbol{\theta}},\mathbf{x})$, estimates (3.13), (3.9), (3.15) and (3.16) can be proved similarly. To obtain (3.9), we divide $N(\hat{\boldsymbol{\theta}},\mathbf{x})$ as in (2.30) and then use (A.3) and (A.5). We omit the details. □

4. L^1 Estimates.

We have the following L^1 error estimates:

THEOREM 4.1. *Consider the model problem for any $n\ge 2$, $1\le d\le n$. When $\Delta\theta$ and h are sufficiently small, we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^1} \le Ch^2\left(\log h^{-1}\right)^\eta, \tag{4.1}$$

$$\|P-P_h\|_{L^1} \le Ch\left(\log h^{-1}\right)^\eta, \tag{4.2}$$

where $C>0$, $\eta\ge 0$ are constants that do not depend on d, h or $\Delta\theta$, but η may vary with n and different assumptions on m and s.

Proof. We only prove (4.1) since the proof for (4.2) is very similar. Throughout this proof we use $C>0$ to denote uniform constants that do not depend on $\hat{\boldsymbol{\theta}}$, h or $\Delta\theta$ but may not remain the same. From (2.27) we have

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{L^1} &= \int_{\mathbb{U}}|\mathbf{u}-\mathbf{u}_h|\,d\mathbf{x} = \int_{\mathbb{U}}\left|\sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta}\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})(\Delta\theta)^d\right|d\mathbf{x}\\ &\le \sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta}\left(\int_{\mathbb{U}}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|d\mathbf{x}\right)(\Delta\theta)^d \le C\max_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta}\int_{\mathbb{U}}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|d\mathbf{x}.\end{aligned} \tag{4.3}$$

For any $\hat{\boldsymbol{\theta}}$, we divide $\mathbb{U}$ as follows:

$$\mathbb{U} = \mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta) \cup \mathbb{A}^c(\hat{\boldsymbol{\theta}}, \Delta\theta), \tag{4.4}$$

$$\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta) = \{\mathbf{x} \in \mathbb{U} : \left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right| \leq \lambda\Delta\theta\}, \tag{4.5}$$

$$\mathbb{A}^c(\hat{\boldsymbol{\theta}}, \Delta\theta) = \{\mathbf{x} \in \mathbb{U} : \left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right| > \lambda\Delta\theta\}, \tag{4.6}$$

where λ is the same constant introduced in Section 3. Then we have

$$\int_{\mathbb{U}} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x} = \int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x} + \int_{\mathbb{A}^c(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x}. \tag{4.7}$$

Further analysis relies on the concrete estimates of the Green's functions and $\mathbf{M}$, N which depend on n. The remaining part of the proof is written for the 2D problem ($n = 2$), but the argument leads to basically the same results for higher dimensional ($n \geq 3$) problems. Consider $\int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x}$ first. Dividing $\mathbf{M}(\hat{\boldsymbol{\theta}})$ as in (2.29), we have

$$\begin{aligned}
&\int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x} \\
\leq& \frac{1}{(\Delta\theta)^d} \int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} \left|\mathbf{G}_{\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})\right| d\boldsymbol{\theta} d\mathbf{x} + \int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\mathcal{I}\mathbf{G}_{h,\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}})\right| d\mathbf{x} \\
\leq& C\left(\frac{1}{(\Delta\theta)^d} \int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} \left|\log\left|\mathbf{x} - \mathbf{X}(\boldsymbol{\theta})\right|\right| d\boldsymbol{\theta} d\mathbf{x} + \int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\max_{\mathbf{y} \in \mathcal{R}_{\mathbf{X}(\hat{\boldsymbol{\theta}})}} \mathbf{G}_{h,\mathbf{x}}(\mathbf{y})\right| d\mathbf{x}\right) \\
\leq& C\left(\int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\log\left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right|\right| d\mathbf{x} + \int_{\mathbb{A}(\hat{\boldsymbol{\theta}}, \Delta\theta)} \log h^{-1} d\mathbf{x}\right) \leq Ch^2 \log h^{-1}.
\end{aligned} \tag{4.8}$$

In the second inequality we used the continuity assumption on $\mathbf{F}$, (A.1) and Lemma 2.1. The third inequality is based on (A.4) and (3.6).

Next we estimate $\int_{\mathbb{A}^c(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x}$. First consider the easier case, that is, when $m \geq 3$, $s \geq 1$. Using (3.11), we get

$$\int_{\mathbb{A}^c(\hat{\boldsymbol{\theta}}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x} \leq C \int_{\mathbb{A}^c(\hat{\boldsymbol{\theta}}, \Delta\theta)} \frac{h^2}{\left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right|^2} d\mathbf{x} \leq Ch^2 \log h^{-1}, \tag{4.9}$$

and this along with (4.8) gives

$$\int_{\mathbb{U}} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| d\mathbf{x} \leq Ch^2 \log h^{-1}, \tag{4.10}$$

for all $\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta$. Hence, in this case we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^1} \leq Ch^2 \log h^{-1}. \tag{4.11}$$

Now consider the most general case, that is, when the assumption is $m \geq 2$, $s \geq 0$. In this case, some estimates of $\mathbf{M}$ and N depend on $|\mathbf{x}_1 - \mathbf{X}_1|$ and $|\mathbf{x}_2 - \mathbf{X}_2|$ instead of the distance $\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|$. We further divide $\mathbb{A}^c(\hat{\boldsymbol{\theta}},\Delta\theta)$:

$$
\begin{aligned}
&\mathbb{A}^c(\hat{\boldsymbol{\theta}},\Delta\theta)=\mathbb{A}_1(\hat{\boldsymbol{\theta}},\Delta\theta)\cup\mathbb{A}_2(\hat{\boldsymbol{\theta}},\Delta\theta),\\
&\mathbb{A}_1(\hat{\boldsymbol{\theta}},\Delta\theta)=\{\mathbf{x}\in\mathbb{A}^c(\hat{\boldsymbol{\theta}},\Delta\theta):|x_1-X_1(\hat{\boldsymbol{\theta}})|\le\frac{\lambda}{\sqrt{2}}\Delta\theta\ \text{ or }\ |x_2-X_2(\hat{\boldsymbol{\theta}})|\le\frac{\lambda}{\sqrt{2}}\Delta\theta\},\\
&\mathbb{A}_2(\hat{\boldsymbol{\theta}},\Delta\theta)=\{\mathbf{x}\in\mathbb{A}^c(\hat{\boldsymbol{\theta}},\Delta\theta):|x_1-X_1(\hat{\boldsymbol{\theta}})|>\frac{\lambda}{\sqrt{2}}\Delta\theta,\ |x_2-X_2(\hat{\boldsymbol{\theta}})|>\frac{\lambda}{\sqrt{2}}\Delta\theta\}.
\end{aligned}
$$

On $\mathbb{A}_1(\hat{\boldsymbol{\theta}},\Delta\theta)$, using (3.7) we get

$$
\int_{\mathbb{A}_1(\hat{\boldsymbol{\theta}},\Delta\theta)}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|d\mathbf{x}\le C\int_{\mathbb{A}_1(\hat{\boldsymbol{\theta}},\Delta\theta)}\frac{h}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|}d\mathbf{x}\le Ch^2\log h^{-1}. \tag{4.12}
$$

On $\mathbb{A}_2(\hat{\boldsymbol{\theta}},\Delta\theta)$, using (3.8) we get

$$
\begin{aligned}
\int_{\mathbb{A}_2(\hat{\boldsymbol{\theta}},\Delta\theta)}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|d\mathbf{x}&\le C\int_{\mathbb{A}_2(\hat{\boldsymbol{\theta}},\Delta\theta)}\frac{h^2\log h^{-1}}{\left|x_1-X_1(\hat{\boldsymbol{\theta}})\right|\cdot\left|x_2-X_2(\hat{\boldsymbol{\theta}})\right|}d\mathbf{x}\\
&\le Ch^2\left(\log h^{-1}\right)^3.
\end{aligned} \tag{4.13}
$$

Adding up (4.8), (4.12) and (4.13), we get

$$
\int_{\mathbb{U}}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|d\mathbf{x}\le Ch^2\left(\log h^{-1}\right)^3, \tag{4.14}
$$

for all $\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta$ and hence in this case ($m\ge 2$, $s\ge 0$) we have

$$
\int_{\mathbb{U}}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|d\mathbf{x}\le Ch^2\left(\log h^{-1}\right)^3. \tag{4.15}
$$

When the assumptions on m and s are between these two cases, we can obtain in a similar way the corresponding L^1 error estimates in the form of (4.1) using the estimates provided in Lemma 3.2, for some $1\le\eta\le 3$. We omit the details. ☐

5. L^p Estimates. In this section, for the model problem we develop L^p error estimates for all $p\ge 1$. Our approach is to first establish L^∞ error estimates and then interpolate them with the L^1 error estimates provided in Section 4. The velocity field and the pressure are only bounded when $n-d$ is either 0 or 1, so we only consider these two situations.

From the assumption of $\mathbf{X}$ being a C^2 diffeomorphism between Γ and Θ and Γ being away from the boundary of $\mathbb{U}$, we claim that we can extend Γ to a slightly bigger set that includes any point $\mathbf{x}\in\mathbb{U}$ that is *sufficiently close* to Γ in the sense that $|\mathbf{x}-\mathbf{X}|\le\sigma$, $\forall\mathbf{X}\in\Gamma$, where $\sigma>0$ is a small constant that only depends on the geometry of Γ. The precise meaning of extending Γ is explained as follows. When $d=n-1$, it means that for a set $\bar{\Theta}=\{(\boldsymbol{\theta},\theta_{d+1}):\boldsymbol{\theta}\in\Theta,\theta_{d+1}\in\mathbf{I}\}$, where $\mathbf{I}\subset\mathbb{R}$ is a closed interval, we can define a function $\bar{\mathbf{X}}$ on $\bar{\Theta}$ such that for some point $\theta_0\in\mathbf{I}^\circ$, where $\mathbf{I}^\circ$ denotes the interior of $\mathbf{I}$, we have $\bar{\mathbf{X}}(\bar{\boldsymbol{\theta}})=\mathbf{X}(\boldsymbol{\theta})$ for all $\bar{\boldsymbol{\theta}}=(\boldsymbol{\theta},\theta_0)\in\bar{\Theta}$. When $d=n$, it means that we can extend $\mathbf{X}$ onto a bigger compact set $\bar{\Theta}\subset\mathbb{R}^n$, $\Theta\subset\subset\bar{\Theta}$ such that the new function $\bar{\mathbf{X}}$ defined on $\bar{\Theta}$ is consistent with $\mathbf{X}$ on Θ. In both cases, the extended immersed structure includes all $\mathbf{x}\in\mathbb{U}$ that is sufficiently close to Γ, in the sense that $|\mathbf{x}-\mathbf{X}| \le \sigma$, $\forall \mathbf{X} \in \Gamma$. Furthermore, $\bar{\mathbf{X}}$ is a C^2 diffeomorphism between $\bar{\Theta}$ and $\bar{\Gamma}$, while $\bar{\Gamma}$ is still away from the boundary of $\mathbb{U}$. We have $\mathbf{x} = \bar{\mathbf{X}}(\bar{\boldsymbol{\theta}}^{\mathbf{x}}) \in \bar{\Gamma} = \mathbf{X}(\bar{\Theta})$, for some unique $\bar{\boldsymbol{\theta}}^{\mathbf{x}} \in \bar{\Theta}$. As a special case, this extension argument is used in [6] to obtain global error estimates for a similar 2D model problem ($n=2$, $d=1$), where the added parameter θ_{d+1} denotes the distance from $\mathbf{x}$ to the immersed boundary. The result in Lemma 3.1 holds for the extended function $\bar{\mathbf{X}}$ on $\bar{\Theta}$ as well and we use the same notation ϵ to denote the corresponding small constant. We set $\epsilon < \sigma$ so that for any $\mathbf{x} \in \mathbb{U}$, as long as $|\mathbf{x}-\mathbf{X}| \le \epsilon$ for some $\mathbf{X} \in \Gamma$, Lemma 3.1 can be applied.

First we state and prove the L^∞ error estimates for both cases, $d = n-1$ and $d = n$. The proofs are only written in detail for $n = 2$, but when $n \ge 3$ the same results can be obtained similarly. As mentioned earlier, when $n = 2$, $d = 1$, the L^∞ estimates for the velocity field have been studied in [6]. Here we rewrite the proof in a more general framework that can be applied to the pressure and higher dimensional problems. In all the proofs throughout this section we use $C > 0$ to denote constants that do not depend on $\Delta\theta$, h, $\mathbf{x}$, $\boldsymbol{\theta}$ or $\hat{\boldsymbol{\theta}}$.

Theorem 5.1. *Consider the model problem when $n \ge 2$, $d = n-1$. When h and $\Delta\theta$ are sufficiently small, we have*

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty} \le Ch(\log h^{-1})^\eta, \tag{5.1}$$

$$\|P-P_h\|_{L^\infty} \le C\left(\log h^{-1}\right)^\eta, \tag{5.2}$$

where $C > 0$ and $\eta \ge 0$ are constants that do not depend on h or $\Delta\theta$ but η may vary with different assumptions on n, m and s.

Proof. We only prove the results for $n = 2$, $d = 1$. For higher dimensional problems, the proof is similar. Consider the velocity field first. From (2.27) we have

$$\|\mathbf{u}-\mathbf{u}_h\|_{L^\infty} = \max_{\mathbf{x}\in\mathbb{U}} |\mathbf{u}(\mathbf{x})-\mathbf{u}_h(\mathbf{x})| \le \max_{\mathbf{x}\in\mathbb{U}} \sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} \left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \Delta\theta. \tag{5.3}$$

As explained earlier in this section, we can extend $\mathbf{X}$ onto a bigger compact set $\bar{\Theta}$. We use the same notation ϵ to denote the small constant and choose $\epsilon < \sigma$. Given that $\Delta\theta$ is sufficiently small, we have $\epsilon > \lambda\Delta\theta$. Divide $\mathcal{G}_\theta$ as follows:

$$\mathcal{G}_\theta = \mathbb{B}_1(\mathbf{x},\Delta\theta) \cup \mathbb{B}_2(\mathbf{x},\Delta\theta) \cup \mathbb{B}_3(\mathbf{x},\Delta\theta), \tag{5.4}$$

$$\mathbb{B}_1(\mathbf{x},\Delta\theta) = \{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta : \left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right| \ge \epsilon\}, \tag{5.5}$$

$$\mathbb{B}_2(\mathbf{x},\Delta\theta) = \{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta : \left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right| \le \lambda\Delta\theta\}, \tag{5.6}$$

$$\mathbb{B}_3(\mathbf{x},\Delta\theta) = \{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta : \lambda\Delta\theta < \left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right| < \epsilon\}. \tag{5.7}$$

For $\mathbb{B}_1(\mathbf{x},\Delta\theta)$, using (3.6) and (3.7) we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_1(\mathbf{x},\Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right| \Delta\theta \le \sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_1(\mathbf{x},\Delta\theta)} \frac{Ch}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|} \Delta\theta \le Ch. \tag{5.8}$$

If $\mathbb{B}_2(\mathbf{x},\Delta\theta) \cup \mathbb{B}_3(\mathbf{x},\Delta\theta) \ne \emptyset$, then $\mathbf{x}$ is sufficiently close to Γ and hence there is a unique $\bar{\boldsymbol{\theta}}^{\mathbf{x}} = (\boldsymbol{\theta}^{\mathbf{x}}, \theta^{\mathbf{x}}_{d+1}) \in \bar{\Theta}$ such that $\mathbf{x} = \bar{\mathbf{X}}(\bar{\boldsymbol{\theta}}^{\mathbf{x}})$. For any $\hat{\boldsymbol{\theta}} \in \mathbb{B}_2(\mathbf{x},\Delta\theta) \cup \mathbb{B}_3(\mathbf{x},\Delta\theta)$, from Lemma 3.1 we see that

$$|\mathbf{x}-\mathbf{X}(\boldsymbol{\theta})| \ge C\,|\boldsymbol{\theta}^{\mathbf{x}}-\boldsymbol{\theta}|\,, \tag{5.9}$$

for all $\boldsymbol{\theta} \in \mathbf{I}_{\hat{\boldsymbol{\theta}}}$. For $\mathbb{B}_2(\mathbf{x}, \Delta\theta)$, similar to (4.8) in the proof of Theorem 4.1, dividing $\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})$ as in (2.29) we get

$$\begin{aligned} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| &\leq \frac{1}{\Delta\theta} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} |\mathbf{G}_{\mathbf{x}}(\mathbf{X}(\boldsymbol{\theta}))\mathbf{F}(\boldsymbol{\theta})| \, d\boldsymbol{\theta} + \left|\mathcal{I}\mathbf{G}_{h,\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}})\right| \\ &\leq \frac{C}{\Delta\theta} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} |\log|\mathbf{x} - \mathbf{X}(\boldsymbol{\theta})|| \, d\boldsymbol{\theta} + C\log h^{-1} \\ &\leq \frac{C}{\Delta\theta} \int_{\mathbf{I}_{\hat{\boldsymbol{\theta}}}} |\log|\boldsymbol{\theta}^{\mathbf{x}} - \boldsymbol{\theta}|| \, d\boldsymbol{\theta} + C\log h^{-1} \leq C\log h^{-1}, \end{aligned} \tag{5.10}$$

where in the second inequality we used Lemma 2.1, (A.1), (A.4), and in the third inequality we used (5.9). Then we have

$$\sum_{\hat{\boldsymbol{\theta}} \in \mathbb{B}_2(\mathbf{x}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| \Delta\theta \leq Ch\log h^{-1}. \tag{5.11}$$

For $\mathbb{B}_3(\mathbf{x}, \Delta\theta)$, using (3.7) and (5.9) we get

$$\sum_{\hat{\boldsymbol{\theta}} \in \mathbb{B}_3(\mathbf{x}, \Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| \Delta\theta \leq C \sum_{\hat{\boldsymbol{\theta}} \in \mathbb{B}^c(\mathbf{x}, \Delta\theta)} \frac{h}{\left|\boldsymbol{\theta}^{\mathbf{x}} - \hat{\boldsymbol{\theta}}\right|} \Delta\theta \leq Ch\log h^{-1}. \tag{5.12}$$

Adding up (5.8), (5.11) and (5.12), we get

$$, \sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| \Delta\theta \leq Ch\log h^{-1}, \tag{5.13}$$

and this along with (5.3) proves

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} \leq Ch\log h^{-1}. \tag{5.14}$$

The pressure P in the continuous problem is globally bounded, that is, $|P(\mathbf{x})| \leq C$ for all $\mathbf{x} \in \mathbb{U}$. Write $P_h(\mathbf{x})$ as in (2.21). We get

$$|P_h(\mathbf{x})| \leq \sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} \left|\mathcal{I}\boldsymbol{\Pi}_{h,\mathbf{x}}(\mathbf{X}(\hat{\boldsymbol{\theta}}))\mathbf{F}(\hat{\boldsymbol{\theta}})\right| \Delta\theta \leq C \sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta} \frac{\log h^{-1}}{\left|\mathbf{x} - \mathbf{X}(\hat{\boldsymbol{\theta}})\right|} \Delta\theta, \tag{5.15}$$

where in the second inequality we use Lemma 2.1, (A.5) and the assumption that $\mathbf{F}$ is C^2. From (5.15), similar to the proof of (5.1), we get

$$|P_h(\mathbf{x})| \leq C\left(\log h^{-1}\right)^2, \text{ and hence } |P(\mathbf{x}) - P_h(\mathbf{x})| \leq C\left(\log h^{-1}\right)^2, \tag{5.16}$$

for any $\mathbf{x} \in \mathbb{U}$. This proves (5.2). ☐

THEOREM 5.2. *Consider the model problem when $n \geq 2$, $d = n$. When $\Delta\theta$ and h are sufficiently small, we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} \leq Ch^2\left(\log h^{-1}\right)^\eta, \tag{5.17}$$

$$\|P - P_h\|_{L^\infty} \leq Ch\left(\log h^{-1}\right)^\eta, \tag{5.18}$$

where $C > 0$ and $\eta \geq 0$ are constants that do not depend on h or $\Delta\theta$ but η may vary with different assumptions on n, m and s.

Proof. We only prove (5.17) for $n = 2$, $d = 2$ since (5.18) and the same results for higher dimensional problems can be proved similarly. From (2.27), we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty} = \max_{\mathbf{x}\in\mathbb{U}} |\mathbf{u}(\mathbf{x}) - \mathbf{u}_h(\mathbf{x})| \le \max_{\mathbf{x}\in\mathbb{U}} \sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| (\Delta\theta)^2, \tag{5.19}$$

and hence it suffices to prove that for any $\mathbf{x} \in \mathbb{U}$, we have

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| (\Delta\theta)^2 \le Ch^2 \left(\log h^{-1}\right)^\eta, \tag{5.20}$$

for some constants $C > 0$ and $\eta \ge 0$ that do not depend on $\mathbf{x}$, h or $\Delta\theta$. Define $\mathbb{B}_1(\mathbf{x}, \Delta\theta)$, $\mathbb{B}_2(\mathbf{x}, \Delta\theta)$ and $\mathbb{B}_3(\mathbf{x}, \Delta\theta)$ as in (5.5), (5.6), and (5.7).

As in the previous theorems, the estimates we are trying to prove depend on the assumption on m and s. We start with the assumption that $m \ge 3$, $s \ge 1$. In this case, the proof is similar to that of Theorem 5.1. For $\mathbb{B}_1(\mathbf{x}, \Delta\theta)$, similar to (5.8), using (3.6) and (3.11) we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_1(\mathbf{x},\Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| (\Delta\theta)^2 \le Ch^2. \tag{5.21}$$

If $\mathbb{B}_2(\mathbf{x}, \Delta\theta) \cup \mathbb{B}_3(\mathbf{x}, \Delta\theta) \ne \emptyset$, then $\mathbf{x}$ is sufficiently close to Γ and hence there is a unique $\boldsymbol{\theta}^{\mathbf{x}} \in \bar{\Theta}$ such that $\mathbf{x} = \mathbf{X}(\boldsymbol{\theta}^{\mathbf{x}})$. Furthermore, for any $\hat{\boldsymbol{\theta}} \in \mathbb{B}_2(\mathbf{x}, \Delta\theta) \cup \mathbb{B}_3(\mathbf{x}, \Delta\theta)$, we have

$$|\mathbf{x} - \mathbf{X}(\boldsymbol{\theta})| \ge C\,|\boldsymbol{\theta}^{\mathbf{x}} - \boldsymbol{\theta}|\,, \tag{5.22}$$

for all $\boldsymbol{\theta} \in \mathbf{I}_{\hat{\boldsymbol{\theta}}}$. For $\mathbb{B}_2(\mathbf{x}, \Delta\theta)$, similar to (5.11), dividing $\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})$ as in (2.29) and using Lemma 2.1, (A.1), (A.4) and (5.22), we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_2(\mathbf{x},\Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| (\Delta\theta)^2 \le Ch^2 \log h^{-1}. \tag{5.23}$$

For $\mathbb{B}_3(\mathbf{x}, \Delta\theta)$, similar to (5.12), using (3.11) and (5.22) we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_3(\mathbf{x},\Delta\theta)} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| (\Delta\theta)^2 \le C \sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}^c(\mathbf{x},\Delta\theta)} \frac{h^2}{\left|\boldsymbol{\theta}^{\mathbf{x}} - \hat{\boldsymbol{\theta}}\right|^2} (\Delta\theta)^2 \le Ch^2 \log h^{-1}. \tag{5.24}$$

Adding up (5.21), (5.23) and (5.24), we get

$$,\sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_\theta} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| (\Delta\theta)^2 \le Ch^2 \log h^{-1}, \tag{5.25}$$

and this is in the form of (5.20).

Now we consider the general case, that is, when the assumption is $m \ge 2$, $s \ge 0$. From Lemma 3.1 we see that for any $\boldsymbol{\theta}_0 \in \Theta$, we have either $\left|\frac{\partial X_1(\boldsymbol{\theta}_0)}{\partial \theta_1}\right| \ge \rho$, $\left|\frac{\partial X_2(\boldsymbol{\theta}_0)}{\partial \theta_2}\right| \ge \rho$ or $\left|\frac{\partial X_1(\boldsymbol{\theta}_0)}{\partial \theta_2}\right| \ge \rho$, $\left|\frac{\partial X_2(\boldsymbol{\theta}_0)}{\partial \theta_1}\right| \ge \rho$. Since $\mathbf{X}$ is C^2 on the compact set Θ, there is a sufficiently small constant $\delta_0 > 0$ such that for all $\boldsymbol{\theta}', \boldsymbol{\gamma}' \in \Theta$, $\left|\boldsymbol{\theta}' - \boldsymbol{\gamma}'\right| \le \delta_0$, we have $\left|\frac{\partial X_i(\boldsymbol{\theta}')}{\partial \theta_j} - \frac{\partial X_i(\boldsymbol{\gamma}')}{\partial \theta_j}\right| \le \frac{\rho}{2}$, for all $i, j = 1, 2$. Pick a sufficiently large integer

$N_0 \geq 2\sqrt{2}\pi/\delta_0$ and evenly divide Θ in both θ_1 and θ_2 directions into $N_0 \times N_0$ closed squares Θ^{ij}, $i, j = 1, 2, \cdots, N_0$. Then for any i, j, we have either

$$\left|\frac{\partial X_1(\boldsymbol{\theta}_0)}{\partial \theta_1}\right| \geq \frac{\rho}{2}, \quad \left|\frac{\partial X_2(\boldsymbol{\theta}_0)}{\partial \theta_2}\right| \geq \frac{\rho}{2}, \quad \text{for all } \boldsymbol{\theta}_0 \in \Theta^{ij}, \tag{5.26}$$

or

$$\left|\frac{\partial X_1(\boldsymbol{\theta}_0)}{\partial \theta_2}\right| \geq \frac{\rho}{2}, \quad \left|\frac{\partial X_2(\boldsymbol{\theta}_0)}{\partial \theta_1}\right| \geq \frac{\rho}{2}, \quad \text{for all } \boldsymbol{\theta}_0 \in \Theta^{ij}. \tag{5.27}$$

Let $\mathcal{G}_\theta^{ij} = \mathcal{G}_\theta \cap \Theta^{ij}$, $i, j = 1, 2, \cdots, N_0$, then $\mathcal{G}_\theta \subset \cup_{i,j=1,\cdots,N_0} \mathcal{G}_\theta^{ij}$. Choose a positive constant $\rho' < \frac{\rho}{2}$. Given that $\Delta\theta$ is sufficiently small, we have either

$$(i). \ \left|\frac{\partial X_1(\boldsymbol{\theta}_0)}{\partial \theta_1}\right| > \rho', \quad \left|\frac{\partial X_2(\boldsymbol{\theta}_0)}{\partial \theta_2}\right| > \rho', \quad \forall\, \boldsymbol{\theta}_0 \in \mathbf{I}_{\hat{\boldsymbol{\theta}}},\ \forall\, \hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta^{ij}, \tag{5.28}$$

or

$$(ii). \ \left|\frac{\partial X_1(\boldsymbol{\theta}_0)}{\partial \theta_2}\right| > \rho', \quad \left|\frac{\partial X_2(\boldsymbol{\theta}_0)}{\partial \theta_1}\right| > \rho', \quad \forall\, \boldsymbol{\theta}_0 \in \mathbf{I}_{\hat{\boldsymbol{\theta}}},\ \forall\, \hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta^{ij}. \tag{5.29}$$

To prove (5.20), it suffices to show that

$$\sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta^{ij}} \left|\mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x})\right| (\Delta\theta)^2 \leq Ch^2 \left(\log h^{-1}\right)^\eta, \tag{5.30}$$

for some constants $C > 0$ and $\eta \geq 0$ that do not depend on i, j, $\mathbf{x}$, h or $\Delta\theta$. Without losing any generality we suppose we are in the case (i) described by (5.28). Otherwise the proof will be similar. Furthermore, define

$$\mathcal{G}_\theta^{ij,1} = \{\hat{\theta} \in \mathcal{G}_\theta^{ij} : \left|x_1 - X_1(\hat{\theta})\right| \geq \left|x_2 - X_2(\hat{\theta})\right|\}, \tag{5.31}$$

$$\mathcal{G}_\theta^{ij,2} = \{\hat{\theta} \in \mathcal{G}_\theta^{ij} : \left|x_1 - X_1(\hat{\theta})\right| \leq \left|x_2 - X_2(\hat{\theta})\right|\}, \tag{5.32}$$

We consider $\mathcal{G}_\theta^{ij,1}$ first. When $\left|x_1 - X_1(\hat{\theta})\right| \geq \left|x_2 - X_2(\hat{\theta})\right|$, we have

$$\left|x_1 - X_1(\hat{\theta})\right| \geq \frac{\sqrt{2}}{2}\left|\mathbf{x} - \mathbf{x}(\hat{\theta})\right|. \tag{5.33}$$

Let P_1, P_2 denote the 2D projections, that is, $P_1(\boldsymbol{\theta}) = \theta_1$, $P_2(\boldsymbol{\theta}) = \theta_2$. From the assumption $\left|\frac{\partial X_2}{\partial \theta_2}\right| > \rho' > 0$ in (5.28), we know that for any fixed $\hat{\theta}_1 \in P_1(\Theta^{ij})$, there exists a unique $\theta_2^{\mathbf{x},\hat{\theta}_1} \in [0, 2\pi]$, $(\hat{\theta}_1, \theta_2^{\mathbf{x},\hat{\theta}_1}) \in \Theta^{ij}$ such that

$$\left|x_2 - X_2(\hat{\theta}_1, \theta_2)\right| \geq \left|X_2(\hat{\theta}_1, \theta_2^{\mathbf{x},\hat{\theta}_1}) - X_2(\hat{\theta}_1, \theta_2)\right| \geq \rho' \left|\theta_2^{\mathbf{x},\hat{\theta}_1} - \theta_2\right|, \tag{5.34}$$

for all $\theta_2 \in [0, 2\pi]$ such that $(\hat{\theta}_1, \theta_2) \in \Theta^{ij}$. Let $\mathbb{B}_l^{ij,1}(\mathbf{x}, \Delta\theta) = \mathbb{B}_l(\mathbf{x}, \Delta\theta) \cap \mathcal{G}_\theta^{ij,1}$, $l = 1, 2, 3$. Further divide $\mathbb{B}_1^{ij,1}(\mathbf{x}, \Delta\theta)$ into the following sets:

$$\mathbb{B}_{1,1}^{ij,1}(\mathbf{x}, \Delta\theta) = \{\hat{\boldsymbol{\theta}} \in \mathbb{B}_1^{ij,1}(\mathbf{x}, \Delta\theta) : |x_2 - X_2(\hat{\boldsymbol{\theta}})| \leq \lambda\Delta\theta\}, \tag{5.35}$$

$$\mathbb{B}_{1,2}^{ij,1}(\mathbf{x}, \Delta\theta) = \{\hat{\boldsymbol{\theta}} \in \mathbb{B}_1^{ij,1}(\mathbf{x}, \Delta\theta) : |x_2 - X_2(\hat{\boldsymbol{\theta}})| > \lambda\Delta\theta\}. \tag{5.36}$$

For $\mathbb{B}_{1,1}^{ij,1}(\mathbf{x},\Delta\theta)$, using (3.6), (3.7) and (5.34) we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{1,1}^{ij,1}(\mathbf{x},\Delta\theta)}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|(\Delta\theta)^2\leq C\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{1,1}^{ij,1}(\mathbf{x},\Delta\theta)}\frac{h}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|}(\Delta\theta)^2\leq Ch^2. \tag{5.37}$$

For $\mathbb{B}_{1,2}^{ij,1}(\mathbf{x},\Delta\theta)$, using (3.6), (3.8), (5.33) and (5.34), we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{1,2}^{ij,1}(\mathbf{x},\Delta\theta)}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|(\Delta\theta)^2\leq C\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{1,2}^{ij,1}(\mathbf{x},\Delta\theta)}\frac{h^2\log h^{-1}}{\left|\theta_2^{\mathbf{x},\hat{\theta}_1}-\hat{\theta}_2\right|}(\Delta\theta)^2\leq Ch^2(\log h^{-1})^2. \tag{5.38}$$

If $\mathbb{B}_{2}^{ij,1}(\mathbf{x},\Delta\theta)\cup\mathbb{B}_{3}^{ij,1}(\mathbf{x},\Delta\theta)\neq\emptyset$, then same as in the previous case ($m\geq 3$, $s\geq 1$), we have (5.22). For $\mathbb{B}_{2}^{ij,1}(\mathbf{x},\Delta\theta)$, following the same argument used to prove (5.23), we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{2}^{ij,1}(\mathbf{x},\Delta\theta)}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|(\Delta\theta)^2\leq Ch^2\log h^{-1}. \tag{5.39}$$

Further divide $\mathbb{B}_{3}^{ij,1}(\mathbf{x},\Delta\theta)$ into the following sets:

$$\mathbb{B}_{3,1}^{ij,1}(\mathbf{x},\Delta\theta)=\{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{3}^{ij,1}(\mathbf{x},\Delta\theta):|x_2-X_2(\hat{\boldsymbol{\theta}})|\leq\lambda\Delta\theta\}, \tag{5.40}$$
$$\mathbb{B}_{3,2}^{ij,1}(\mathbf{x},\Delta\theta)=\{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{3}^{ij,1}(\mathbf{x},\Delta\theta):|x_2-X_2(\hat{\boldsymbol{\theta}})|>\lambda\Delta\theta\}. \tag{5.41}$$

For $\mathbb{B}_{3,1}^{ij,1}(\mathbf{x},\Delta\theta)$, using (3.6), (3.7) and (5.34) we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{3,1}^{ij,1}(\mathbf{x},\Delta\theta)}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|(\Delta\theta)^2\leq C\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{3,1}^{ij,1}(\mathbf{x},\Delta\theta)}\frac{h}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|}(\Delta\theta)^2\leq Ch^2\log h^{-1}. \tag{5.42}$$

For $\mathbb{B}_{3,2}^{ij,1}(\mathbf{x},\Delta\theta)$, using (3.6), (3.8), (5.33), (5.34) and (5.22), we get

$$\begin{aligned}\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{1,2}^{ij,1}(\mathbf{x},\Delta\theta)}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|(\Delta\theta)^2&\leq C\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{1,2}^{ij,1}(\mathbf{x},\Delta\theta)}\frac{h^2\log h^{-1}}{\left|\mathbf{x}-\mathbf{X}(\hat{\boldsymbol{\theta}})\right|\cdot\left|x_2-X_2(\hat{\boldsymbol{\theta}})\right|}(\Delta\theta)^2\\ \leq C&\sum_{\hat{\boldsymbol{\theta}}\in\mathbb{B}_{1,2}^{ij,1}(\mathbf{x},\Delta\theta)}\frac{h^2\log h^{-1}}{\left|\boldsymbol{\theta}^{\mathbf{x}}-\hat{\boldsymbol{\theta}}\right|\cdot\left|\theta_2^{\mathbf{x},\hat{\theta}_1}-\hat{\theta}_2\right|}(\Delta\theta)^2\leq Ch^2(\log h^{-1})^3.\end{aligned} \tag{5.43}$$

Adding up (5.37), (5.38), (5.39), (5.42), (5.43), we get

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_{\theta}^{ij,1}}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|(\Delta\theta)^2\leq Ch^2\left(\log h^{-1}\right)^3. \tag{5.44}$$

Similarly, we can show that

$$\sum_{\hat{\boldsymbol{\theta}}\in\mathcal{G}_{\theta}^{ij,2}}\left|\mathbf{M}(\hat{\boldsymbol{\theta}},\mathbf{x})\right|(\Delta\theta)^2\leq Ch^2\left(\log h^{-1}\right)^3. \tag{5.45}$$

Adding up (5.44) and (5.45), we obtain

$$\sum_{\hat{\boldsymbol{\theta}} \in \mathcal{G}_\theta^{ij}} \left| \mathbf{M}(\hat{\boldsymbol{\theta}}, \mathbf{x}) \right| (\Delta\theta)^2 \le C h^2 \left(\log h^{-1}\right)^3. \tag{5.46}$$

This is in the form of (5.30). For other intermediate cases, the proof is very similar. We omit the details. □

Interpolating the L^1 error estimates from Theorem 4.1 with the L^∞ error estimates from Theorem 5.1 for the case $d = n-1$ and Theorem 5.2 for the case $d = n$ respectively, we obtain the general L^p $(1 \le p \le \infty)$ error estimates for both cases. We state these results as follows.

THEOREM 5.3. *Consider the model problem when $n \ge 2$, $d = n-1$. When $\Delta\theta$ and h are sufficiently small, we have*

$$\left\|\mathbf{u} - \mathbf{u}_h\right\|_{L^p} \le C h^{1+\frac{1}{p}} \left(\log h^{-1}\right)^\eta, \tag{5.47}$$

$$\left\|p - p_h\right\|_{L^p} \le C h^{\frac{1}{p}} \left(\log h^{-1}\right)^\eta, \tag{5.48}$$

where $C > 0$ and $\eta \ge 0$ are constants that do not depend on h or $\Delta\theta$ but η may vary with different assumptions on m and s.

THEOREM 5.4. *Consider the model problem when $n \ge 2$, $d = n-1$. When $\Delta\theta$ and h are sufficiently small, we have*

$$\left\|\mathbf{u} - \mathbf{u}_h\right\|_{L^p} \le C h^2 \left(\log h^{-1}\right)^\eta, \tag{5.49}$$

$$\left\|p - p_h\right\|_{L^p} \le C h^1 \left(\log h^{-1}\right)^\eta, \tag{5.50}$$

where $C > 0$ and $\eta \ge 0$ are constants that do not depend on h or $\Delta\theta$ but η may vary with different assumptions on m and s.

6. Numerical Simulation. We tested the theorems with numerical experiments when $n = 2$, $d = 1, 2$. In this section we provide the results and compare them with the predicted asymptotic error rates.

Consider the model problem described in Section 2.1 when $n = 2$, $d = 1, 2$. Pick a large integer $N > 0$ and let $h = \frac{2\pi}{N}$, $\Delta\theta = \frac{4\pi}{N}$ so that h and $\Delta\theta$ are proportional to each other. Use spectral schemes for Δ_h and ∇_h in (2.4), (2.5) and (2.6) and compute $\mathbf{u}_h$, P_h. It is hard to capture the order of logarithmic terms such as $\log h^{-1}$ with the presence of $o(h^r)$ $(r > 0)$ terms, and hence we only compute the latter. We use the following formulas to compute the L^p $(1 \le p \le \infty)$ error rates for the velocity field and the pressure respectively:

$$\rho_{\mathbf{u}}^p = \log_2 \left(\frac{\left\| \mathbf{u}_h(\mathbf{x}) - \mathbf{u}_{\frac{h}{2}}(\mathbf{x}) \right\|_{L^p}}{\left\| \mathbf{u}_{\frac{h}{2}}(\mathbf{x}) - \mathbf{u}_{\frac{h}{4}}(\mathbf{x}) \right\|_{L^p}} \right), \quad \rho_P^p = \log_2 \left(\frac{\left\| P_h(\mathbf{x}) - P_{\frac{h}{2}}(\mathbf{x}) \right\|_{L^p}}{\left\| P_{\frac{h}{2}}(\mathbf{x}) - P_{\frac{h}{4}}(\mathbf{x}) \right\|_{L^p}} \right). \tag{6.1}$$

We tested various choices of $\mathbf{X}$ and the force functions $\mathbf{F}$ with different discrete delta functions. We chose $p = 1, 2, \infty$. Let $R_{\mathbf{u}}^p$ and R_P^p denote the predicted error rates for the velocity field and the pressure respectively.

When $n = 2$, $d = 1$, to generate Table 6.1 we chose $N = 512$ and

$$\mathbf{X}(\theta) = \frac{\pi}{12} \begin{pmatrix} 12 + \cos\theta(6 + \cos 3\theta) \\ 12 + \sin\theta(6 + \cos 3\theta) \end{pmatrix}, \quad \mathbf{F}(\theta) = \begin{pmatrix} 1 + \sin\theta \\ 1 + \cos\theta \end{pmatrix}. \tag{6.2}$$

(m,s,ss)	$\rho^1_{\mathbf{u}}$	$\rho^2_{\mathbf{u}}$	$\rho^\infty_{\mathbf{u}}$	ρ^1_P	ρ^2_P	ρ^∞_P
$(2,0,0)$	1.9786	1.5240	1.3091	0.9080	0.4907	0.0928
$(2,1,0)$	1.9689	1.4973	1.1772	1.0113	0.4972	0.1757
$(2,0,1)$	1.9792	1.5053	1.2214	0.9316	0.4901	0.0899
$(2,1,1)$	1.9747	1.4906	1.0330	1.0011	0.4965	-0.0245
$(4,0,0)$	1.9798	1.5228	1.2898	0.9106	0.4850	0.0610
$(4,1,0)$	2.0122	1.5095	1.1747	0.9875	0.4755	0.0453
$(4,2,0)$	2.0002	1.5067	1.1802	0.9863	0.4861	0.0539
$(4,3,0)$	1.9878	1.4928	1.1260	1.0010	0.4957	-0.0253
$(4,0,1)$	1.9956	1.5148	1.2515	0.9287	0.4770	0.1040
$(4,3,1)$	1.9806	1.4874	1.0947	1.0008	0.4962	-0.0445
$(6,0,0)$	1.9792	1.5214	1.2890	0.9094	0.4808	0.0516
$(6,5,0)$	1.9841	1.4919	1.0953	0.9989	0.4945	-0.0504
$(6,0,1)$	1.9976	1.5168	1.2574	0.9286	0.4736	0.1098
$(6,5,1)$	1.9804	1.4885	1.0762	1.0005	0.4953	-0.0461

TABLE 6.1

Asymptotic error rates for the example problem (6.2) in which $n = 2$, $d = 1$. m: Moment order. s: Smoothing order. ss: Indicator of whether condition (2.11) is satisfied. $\rho^p_{\mathbf{u}}$: Computed L^p error rates for the velocity field. ρ^p_P: Computed L^p error rates for the pressure.

The computed error rates $\rho^1_{\mathbf{u}}$, $\rho^2_{\mathbf{u}}$, $\rho^\infty_{\mathbf{u}}$, ρ^1_P, ρ^2_P, ρ^∞_P are very close to the predicted results $R^1_{\mathbf{u}} = 2$, $R^2_{\mathbf{u}} = 1.5$, $R^\infty_{\mathbf{u}} = 1$, $R^1_P = 1$, $R^2_P = 0.5$, $R^\infty_P = 0$.

When $n = 2$, $d = 2$, to generate Table 6.2 we chose $N = 512$ and

$$\mathbf{X}(\theta) = \frac{1}{6}\begin{pmatrix} 6\pi + (r+\pi)\cos\theta \\ 4\pi + (r+\pi)\sin\theta \end{pmatrix}, \quad \mathbf{F}(\theta) = \begin{pmatrix} 1 + r\sin\theta \\ 2 + r\cos\theta \end{pmatrix}. \tag{6.3}$$

The computed error rates $\rho^1_{\mathbf{u}}$, $\rho^2_{\mathbf{u}}$, $\rho^\infty_{\mathbf{u}}$, ρ^1_P, ρ^2_P, ρ^∞_P are very close to the predicted results $R^1_{\mathbf{u}} = R^2_{\mathbf{u}} = R^\infty_{\mathbf{u}} = 2$, $R^1_P = R^2_P = R^\infty_P = 1$.

7. Concluding Remarks. In this paper, we established L^p $(1 \le p \le \infty)$ error estimates in terms of $h^r(\log h^{-1})^\eta$ for a prototypical model problem described in Section 2.1, using results from [5]. When $p = 1$, r is independent of n and d. When $p > 1$, the estimates only exist when $d = n-1, n$ and r only depends on $n - d$. We tested the theorems with numerical experiments and the computed results suggest the predicted error rates are optimal.

Some technical assumptions are made to streamline the presentation. As seen in the proofs here and in [5], the error rates depend on m, s, q and the regularity of $\mathbf{X}$ and $\mathbf{F}$, where q is the order of the discretization schemes Δ_h and ∇_h. We are mainly interested in the impact of m and s on the error rates, and for this reason we assume $\mathbf{X}$ and $\mathbf{F}$ are C^2 and use spectral schemes for Δ_h and ∇_h so that $q = \infty$. The C^2 assumption on $\mathbf{X}$ and $\mathbf{F}$ is used to get second order accuracy for the mid-point integration scheme. In practice we usually use second order accurate schemes for Δ_h and ∇_h, i.e., $q \ge 2$ so that we do not lose too much accuracy from discretizing the spatial derivatives. In fact, from the proofs we see that as long as $q \ge 2$, it will not have any impact on on the error rate r, despite that η may be different depending on whether $q \ge 3$. The assumption $m \ge 2$ guarantees that the immersed boundary error is of second order. In some cases where we only need first order accuracy, some weaker assumptions may be sufficient, such as $q \ge 1$, $m \ge 1$, and $\mathbf{X}$, $\mathbf{F}$ being C^1.

(m,s,ss)	$\rho_{\mathbf{u}}^1$	$\rho_{\mathbf{u}}^2$	$\rho_{\mathbf{u}}^\infty$	ρ_P^1	ρ_P^2	ρ_P^∞
$(2,0,0)$	2.0049	1.9994	1.9320	1.0223	0.9954	0.9744
$(2,1,0)$	1.9974	1.9957	1.9473	1.0338	1.0013	0.8984
$(2,0,1)$	1.9996	1.9964	1.9158	1.0225	0.9954	0.9135
$(2,1,1)$	1.9969	1.9958	1.9526	1.0478	1.0013	0.9385
$(4,0,0)$	2.0238	2.0025	1.9228	1.0194	0.9956	0.9530
$(4,1,0)$	2.0295	1.9992	1.9101	1.0175	0.9939	0.9572
$(4,2,0)$	2.0381	1.9961	1.8869	1.0241	0.9846	0.9442
$(4,3,0)$	2.0567	2.0006	1.8512	1.0477	0.9940	0.9529
$(4,0,1)$	2.0269	2.0009	1.9302	1.0166	0.9944	0.9515
$(4,3,1)$	2.0701	2.0067	1.8430	1.0579	0.9931	0.9572
$(6,0,0)$	2.0248	2.0031	1.9216	1.0192	0.9961	0.9531
$(6,5,0)$	2.0685	1.9952	1.8255	1.0696	0.9939	0.9227
$(6,0,1)$	2.0271	2.0022	1.9335	1.0156	0.9947	0.9638
$(6,5,1)$	2.0787	1.9983	1.8205	1.0837	0.9961	0.9164

Table 6.2

Asymptotic error rates for the example problem (6.3) in which $n = 2$, $d = 2$. m: Moment order. s: Smoothing order. ss: Indicator of whether condition (2.11) is satisfied. $\rho_{\mathbf{u}}^p$: Computed L^p error rates for the velocity field. ρ_P^p: Computed L^p error rates for the pressure.

It is worth pointing out the assumption that h and $\Delta\theta$ are proportional to each other is not necessary for either point-wise or L^p convergence. For convergence at the points that are not on Γ, h and $\Delta\theta$ can be chosen independent of each other. For L^p convergence, h and $\Delta\theta$ can also be basically independent of each other, as long as $\log(\Delta\theta)^{-1} = o(h^{\epsilon_1})$ and $\log h^{-1} = o((\Delta\theta)^{\epsilon_2})$ for some constant $\epsilon_1, \epsilon_2 > 0$. Both the point-wise and the L^p error estimates may vary depending on the relationship between h and $\Delta\theta$. In most cases, it is when h and $\Delta\theta$ are proportional to each other that we obtain the highest error rates. Depending on the problem, when h and $\Delta\theta$ are not chosen to be proportional to each other, similar error estimates can be obtained by using the argument presented in this paper.

Appendix A. Estimates of Green's Functions. In this appendix we provide some estimates of the Green's functions $\mathbf{G}$, $\mathbf{\Pi}$ and their derivatives when $n = 2$. We also state some estimates of the discrete Green's functions $\mathbf{G}_h$, $\mathbf{\Pi}_h$ when the spectral scheme without filtering is used to discretize the spatial derivatives. The results for the velocity field have been proved in [6] and for the pressure similar results can be obtained. We only state the results.

Lemma A.1. *When $n = 2$, for any $\mathbf{x} \in \mathbb{U}$, $|\mathbf{x}| \neq 0$ we have*

$$|G_{ij}(\mathbf{x})| \le C \log |\mathbf{x}|^{-1}, \tag{A.1}$$

$$\left| \frac{\partial^{\alpha_1+\alpha_2} G_{ij}(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right| \le C \frac{1}{|\mathbf{x}|^{\alpha_1+\alpha_2}}, \quad if \quad \alpha_1+\alpha_2 > 0, \tag{A.2}$$

$$\left| \frac{\partial^{\alpha_1+\alpha_2} \Pi_{ij}(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right| \le C \frac{1}{|\mathbf{x}|^{\alpha_1+\alpha_2+1}}, \tag{A.3}$$

where $i,j = 1,2$, G_{ij} and Π_{ij} denote the $(ij)^{th}$ components of $\mathbf{G}$ and $\mathbf{\Pi}$ respectively, α_i and α_j are nonnegative integers and $C > 0$ are constants.

Lemma A.2. *When $n = 2$, for any $\mathbf{x} \in \mathbb{U}$, $|\mathbf{x}| \neq 0$, we have*

$$|G_{h,ij}(\mathbf{x})| \leq C \log h^{-1}, \tag{A.4}$$

$$|\Pi_{h,ij}(\mathbf{x})| \leq C\frac{1}{h}, \quad |\Pi_{h,ij}(\mathbf{x})| \leq C\frac{\log h^{-1}}{|\mathbf{x}|}, \tag{A.5}$$

where $i, j = 1, 2$, G_{ij} and Π_{ij} denote the $(ij)^{th}$ components of $\mathbf{G}_h$ and $\mathbf{\Pi}_h$ respectively and $C > 0$ are constants.

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